On the Cartesian product of non well-covered graphs

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Abstract

A graph is well-covered if every maximal independent set has the same cardinality, namely the vertex independence number. We answer a question of Topp and Volkmann (1992) and prove that if the Cartesian product of two graphs is well-covered, then at least one of them must be well-covered.

Keywords: maximal independent set; well-covered; Cartesian product

1 Introduction

A *well-covered graph* G (Plummer [3]) is one in which every maximal independent set of vertices has the same cardinality. That is, every maximal independent set (equivalently, every independent dominating set) is a maximum independent set. This class of graphs has been investigated by many researchers from several different points of view. Among these are attempts to characterize those well-covered graphs with a girth or a maximum degree restriction. For more details on these approaches as well as others see the surveys by Plummer [4] and by Hartnell [2].

Topp and Volkmann [5] investigated how the standard graph products interact with the class of well-covered graphs. They asked the following question that was restated by Fradkin [1].

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Question 1. ([5]) Do there exist non well-covered graphs whose Cartesian product is well-covered?

The principal result of this paper is the following theorem that answers Question 1 in the negative.

Theorem 2. *If G and H are graphs whose Cartesian product is well-covered, then at least one of G or H is well-covered.*

In Section 2 we define the terms used most often in this paper; standard graph theory terminology is used throughout. We then establish Theorem 2 in Section 3.

2 Definitions

If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are any two graphs, the *Cartesian product* of G_1 and G_2 is the graph denoted $G_1 \square G_2$ whose vertex set is the Cartesian product of their vertex sets $V_1 \times V_2$. Two vertices (x_1, x_2) and (y_1, y_2) are adjacent in $G_1 \square G_2$ if either $x_1 = y_1$ and $x_2y_2 \in E_2$, or $x_1y_1 \in E_1$ and $x_2 = y_2$. Note that if I_1 is independent in G_1 and I_2 is independent in G_2, then the set $I_1 \times I_2$ is independent in $G_1 \square G_2$.

For an arbitrary graph G we follow Fradkin [1] and define a *greedy independent decomposition* of G to be a partition $A_1, A_2, \ldots, A_t$ of $V(G)$ such that A_1 is a maximal independent set in G, and for each $2 \leqslant i \leqslant t$, the set A_i is a maximal independent set in the graph $G - (A_1 \cup \cdots \cup A_{i-1})$. One way to construct maximal independent sets in the Cartesian product $G \square H$ is to select any greedy independent decomposition $A_1, A_2, \ldots, A_t$ of G and an arbitrary greedy independent decomposition $B_1, B_2, \ldots, B_s$ of H and combine them into what is called a "diagonal" set of the product as $M = \cup_i(A_i \times B_i)$. If $s \neq t$, then there are as many sets in this union as the smaller of s and t.

For a vertex x of a graph G, the *open neighborhood* of x is the set $N(x)$ defined by $N(x) = \{w \in V(G) \,|\, xw \in E(G)\}$, and the *closed neighborhood*, $N[x]$, of x is the set $N(x) \cup \{x\}$. For $A \subseteq V(G)$ we define $N(A)$ to be $\cup_{x \in A} N(x)$ and $N[A] = N(A) \cup A$. The *vertex independence number* of a graph G is the cardinality of a largest independent set in G. We denote the vertex independence number of G by $\alpha(G)$ and refer to an independent set of this order as an $\alpha(G)$-set. If a graph G has an independent set M such that $G - N[M] = \{x\}$ for some vertex x, then x is said to be an *isolatable vertex* of G. The existence of such a vertex is central to our work.

Lemma 3. *Let G be a graph in which no vertex is isolatable. If I is any maximum independent set in G and x is any vertex of I, $G - N[I - \{x\}]$ is a clique of order at least two.*

Proof. Suppose I is an $\alpha(G)$-set and that I has a vertex v such that the graph $G - N[I - \{v\}]$ has an independent set A of size at least two. Then $I' = (I - \{v\}) \cup A$ is independent in G and has order larger than $|I| = \alpha(G)$, a contradiction. Therefore, $G - N[I - \{v\}]$ is a clique. Since G has no isolatable vertex it follows that $G - N[I - \{v\}]$ has order at least two. □

3 Main Results

We first reduce the study of when a Cartesian product is well-covered by considering the existence of isolatable vertices in the two factors.

Theorem 4. *Suppose that H is not well-covered and G has an isolatable vertex. Then $G \,\square\, H$ is not well-covered.*

Proof. Let A and B be maximal independent subsets of H with $|A| > |B|$, and suppose that x is an isolatable vertex in G. Let I be an independent set in G such that x is an isolated vertex in the graph $G - N[I]$. Extend the independent set $I \times A$ to a maximal independent set J of $(G - N[x]) \,\square\, H$. Let $m = |J|$. Note that J dominates $N_G(x) \times A$ (and perhaps other vertices of $N_G(x) \times V(H)$), but J does not contain any vertices from $N_G[x] \times V(H)$.

Let $J_1 = J \cup (\{x\} \times A)$ and $J_2 = J \cup (\{x\} \times B)$. By the choice of A and B it is clear that $|J_1| > |J_2|$. Let X_A denote the set of vertices in $N_G(x) \times V(H)$ that are not dominated by J_1. Similarly, let X_B denote the set of vertices in $N_G(x) \times V(H)$ that are not dominated by J_2. Note that any vertex in $N_G(x) \times V(H)$ that is dominated by J_1 is also dominated by J. Since $J \subset J_2$, it follows that if a vertex of $N_G(x) \times V(H)$ is dominated by J_1 it is also dominated by J_2. Hence, the set X_B is a subset of X_A.

Choose a maximal independent set L of the subgraph of $G \,\square\, H$ induced by X_B. Then $J_2 \cup L$ is a maximal independent set in $G \,\square\, H$. Extend L to a maximal independent set M of the subgraph of $G \,\square\, H$ induced by X_A. Now, $J_1 \cup M$ is a maximal independent set of $G \,\square\, H$, and

$$|J_1 \cup M| = |J_1| + |M| > |J_2| + |M| \geqslant |J_2| + |L| = |J_2 \cup L|\,.$$

Therefore, $G \,\square\, H$ has maximal independent sets of distinct cardinalities, and thus $G \,\square\, H$ is not well-covered. □

It now follows that if both of G and H are not well-covered but $G \,\square\, H$ is well-covered, then neither G nor H has an isolatable vertex.

Lemma 5. *Let G and H be graphs such that neither has an isolatable vertex. If $G \,\square\, H$ is well-covered, then both G and H have the property that if M is any maximal independent set of the graph, that graph must have a maximal independent set N that is disjoint from M. Furthermore, at least one of G or H has the property that any two disjoint maximal independent sets have the same cardinality.*

Proof. As stated above, neither G nor H has an isolatable vertex. Let $I_1, I_2, \ldots, I_t$ be any greedy independent decomposition of G and $J_1, J_2, \ldots, J_s$ be any greedy independent decomposition of H. Let $p = \min\{s, t\}$. The (so-called "diagonal") set $M = (I_1 \times J_1) \cup (I_2 \times J_2) \cup \cdots \cup (I_p \times J_p)$ is maximal independent in $G \,\square\, H$. Since $G \,\square\, H$ is well-covered, this implies that $\alpha(G \,\square\, H) = |M| = \sum_{k=1}^{p} |I_k| \cdot |J_k|$.

Since J_1 is an independent set in H and J_2 is a maximal independent set in $H - J_1$, if there exists a vertex $u \in J_1$ that is not dominated by J_2, then u is isolated in $H - N[J_2]$.

(This follows since every neighbor of u in H would thus belong to $V(H) - (J_1 \cup J_2)$, and none of these remains in $H - N[J_2]$.) This contradicts the fact that H does not have an isolatable vertex. Therefore, J_2 is actually a maximal independent set in H as well as in $H - J_1$. By an identical argument it follows that I_2 is a maximal independent set in G. Suppose that $a = |J_1|$ and $b = |J_2|$ and that $a \neq b$. Let $c = |I_1|$ and $d = |I_2|$. Since I_1 and I_2 are disjoint maximal independent sets in G, the list $I_2, I_1, I_3, \ldots, I_t$ is also a greedy independent decomposition of G. This implies

$$ca + db + \sum_{k=3}^{p} |I_k| \cdot |J_k| = \alpha(G \,\Box\, H) = da + cb + \sum_{k=3}^{p} |I_k| \cdot |J_k| \,,$$

since $G \,\Box\, H$ is well-covered, and thus $ca + db = da + cb$. Since $a \neq b$ we get $c = d$; that is, $|I_1| = |I_2|$. Since $I_1, I_2, \ldots, I_t$ is an arbitrary greedy independent decomposition of G, the lemma follows. □

We now proceed to prove our main result.

Theorem 2. *If G and H are graphs such that $G \,\Box\, H$ is well-covered, then at least one of G or H is well-covered.*

Proof. Suppose by way of contradiction that the statement is not true. Let G and H be a pair of graphs neither of which is well-covered but such that $G \,\Box\, H$ is well-covered. As above we may assume that no vertex of either G or H can be isolated in its own graph. From Lemma 5 we may assume without loss of generality that G has the property that any two maximal independent sets of different cardinalities must intersect nontrivially.

Since G is not well-covered, there exists a maximal independent set whose cardinality is less than $\alpha(G)$. From the collection of all maximal independent sets in G choose a pair, say I and J, such that $|J| < |I| = \alpha(G)$ and $|I \cap J|$ is as small as possible. Since $|I| \neq |J|$ there exists $v \in I \cap J$. Let $F = G - N[I - \{v\}]$. By Lemma 3 this subgraph F is a clique of order at least two. Let w be any vertex of F such that $w \neq v$, and let $I' = (I - \{v\}) \cup \{w\}$. Note that I' is independent, $|I'| = |I|$, and yet $|I' \cap J| = |I \cap J| - 1$ contradicting our choice of I and J. Therefore, G is well-covered. □

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