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THE RUN OF SUPERADIABATICITY IN STELLAR CONVECTION ZONES. I. THE SUN

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ABSTRACT

The effect of adjustments to the superadiabatic layer (SAL) in a model of the Sun on the *p*-mode oscillation frequencies has been studied. Numerical simulations of solar convection by Kim and coworkers have shown that the usual mixing length approximation (MLA) overestimates the convective efficiency in the SAL. To correct for the overestimated convective efficiency in the calculation of the temperature gradient and the mean structure of the SAL, we have adopted a variable mixing length parameter, which decreases as the surface is approached, based on a simple parameterization suggested by the simulations. We find that these changes to the structure of the SAL reduce the discrepancies between observed and calculated oscillation frequencies in the low to intermediate *l* range.

Subject headings: convection — Sun: interior — Sun: oscillations

1. INTRODUCTION

The progress of helioseismology has led to increased interest in probing the structure of the strongly superadiabatic layer (SAL) that forms the transition between the deep layers of the solar convection zone, where the superadiabaticity of the temperature gradient, $\nabla - \nabla_{\rm ad}$, is only of the order of one part in one million, and the outer optically thin atmospheric layers of the Sun, where radiative transfer dominates ($\nabla - \nabla_{\rm ad}$ is of the order unity). Current solar models and convection simulations show that this layer is probably very thin in the Sun, extending from the surface to ~ 300 km below the surface. It is believed that the SAL is the site of the excitation of the *p*-mode oscillations and that the SAL structure determines the extent of convective overshoot near the surface. In standard solar models, the mean structure of the SAL is determined by the mixing length approximation (MLA), in which the mixing length is a free parameter, constrained primarily by the observed radius and to a much lesser degree by the luminosity of the Sun. If we could model this layer correctly through a complete understanding of the interaction of radiation and convection, then we could determine the radius of solar models from first principles, without recourse to any adjustable parameter, such as the mixing length parameter.

The solar *p*-mode frequencies, particularly the higher frequency modes, are sensitive to the detailed structure of the SAL. While many properties of the *p*-modes themselves can be interpreted in terms of standard physical models of the solar interior, it is increasingly evident that the outer layers must be better understood if theoretical models are to predict the observed frequencies more precisely. There are indications that at least part of the problem lies in the SAL, and that the standard treatment of stellar convection based on the MLA leads to an underestimate of the efficiency of radiative transfer in the shallower part of the SAL. Two areas of refinement that could improve the calculated values of the *p*-mode frequencies are (1) the inclusion of nonadiabatic effects from radiation and turbulence in the pulsation calculations; and (2) the adoption of a more sophisticated calculation of the structure and thermodynamics of the SAL.

Using a basic theoretical model of turbulence Balmforth (1992a, 1992b, 1992c) has studied the effects of turbulence on the structure of the solar model and on the *p*-mode oscillation calculation. The effects of radiative dissipation on the *p*-mode oscillation calculation have been explored by Ando & Osaki (1975), Antia, Chitre, & Narasimha (1982), Christensen-Dalsgaard & Frandsen (1983), Cox, Guzik, & Kidman (1989), Guenther (1994), and Guzik & Cox (1993). Uncertainties in the structure of the outer layers of the model probably also arise from uncertainties in the low-temperature opacities, the atmosphere model, and the surface boundary conditions.

In this paper we will modify the structure of the SAL according to the suggestion from numerical simulations of convection and will study the effect of the modified SAL on the predicted *p*-mode frequencies in the Sun. Others have pursued similar investigations. Monteiro, Christensen-Dalsgaard, & Thompson (1995) have explored different parameterizations of the SAL structure as constrained by inversion of the observed solar *p*-modes. Rosenthal et al. (1995) have investigated the structural effects of the mean turbulent pressure derived from models of a nonlocal mixing length theory (Gough 1977; Balmforth 1992a), or from the numerical simulations of convection of Stein & Nordlund (1989). By contrast, in this paper, the structure of the SAL is more accurately reproduced by modifying the MLA treatment of convection in our solar models, using recent numerical simulations of the shallow part of the solar convection zone as a guide (Kim 1993; Kim et al. 1995, 1996a, 1996b). The Kim simulations include a physically realistic treatment of the local microscopic physics, from the stellar evolution code YREC (see Guenther et al. 1992), and a treatment of the coupling betwen radiative and convective

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energy transports in the diffusion approximation. We note that because the simulation describes radiative transfer in the diffusion approximation, it becomes invalid in the optically thin regions of the atmosphere. But the simulations provide a rough quantitative estimate of the departure of the predicted structure from the structure prescribed by the MLA in the shallow convective layers.

A simple modification of the MLA cannot provide a realistic description of the dynamics of convection. Kim's simulations differ from the MLA in the very fundamental respect that turbulent convection, including the interaction between convection and radiation, and ionization were explicitly treated. In this paper *we shall assume that most of the structural features of the SAL can be modeled using the MLA formalism with a mixing length that varies with depth.* We show that this approach can remove most of the discrepancy between the observed and calculated *p*-mode frequencies and thus must provide a good approximation to the mean run of the speed of sound in the outer layers of the Sun. Importantly, even though we have modified the form of the MLA, the mean stratification of the model itself still satisfies the usual constraints of stellar models relating to energy transfer, thermodynamics, and hydrostatic equilibrium. As a result, these still exploratory models already provide us with the means of reconstructing the run of the mean temperature gradient in the solar atmosphere and subphotospheric layers in a much more realistic way than the MLA. This confirms the direction for the improvement of the SAL suggested by the simulations.

2. TOWARD AN UNDERSTANDING OF SOLAR CONVECTION

Before describing our approach, it is useful to situate it in the context of other recent research on solar convection, all addressing the issue of the SAL from a different point of view. Note that in this paper we are concerned only with the problem of the mean hydrostatic structure of the SAL layer and not the detailed dynamics of convection in the Sun.

2.1. *The Mixing Length Approximation*

All standard solar models use the MLA to describe convection in the outer layers of the Sun. The MLA combines a phenomenological description of convection based on the notion that, for all practical purposes, the convective energy is carried by convective elements of a single size that ultimately dissolve into their surroundings via radiative diffusion (Vitense 1953; Böhm-Vitense 1958; Gough & Weiss 1976; Baker & Gough 1979). We note that the radiative diffusion approximation used in the MLA is valid only in the deep layers and breaks down in the optically thin surface layers.

The MLA correctly predicts a very small superadiabaticity in the deep convective layers. This means that the temperature gradient, for the purpose of calculating the envelope structure and radius, is equal to the local adiabatic temperature gradient. In constructing the models, the height of the superadiabatic peak depends on the details of the local physics (primarily the local equation of state and opacities) and the choice of the mixing length parameter—all of which control the efficiency of convection in the SAL. In a laboratory fluid, the effective mixing length is usually found to equal the size of the convective region. In stellar models, where fluids are compressible, and convection can extend over many pressure (or density) scale heights, the mixing length is usually set equal to a fraction α of the local pressure scale height. The choice of α (mixing length parameter), which is effectively a measure of the efficiency of convection in the SAL, determines the local temperature gradient in the SAL.

The choice of α also affects the calculated model radius. We emphasize here that because the SAL is itself thin compared to the total radius ($\sim 0.04\%$), its thickness is not a significant factor in determining the radius (at least for stars near the main sequence). But the precise structure of the SAL, specifically the run of the thermodynamic variables in the region which approaches adiabatic equilibrium just below the superadiabatic peak, determines the specific entropy in the adiabatic envelope. It has long been known that in a stellar model with a convective adiabatic envelope, the depth of the convective envelope is a function of its specific entropy (Schwarzschild 1958; for a more general discussion see also Larson 1974). It is therefore the sensitivity to α of the specific entropy in the deep adiabatic layers of the convective envelope that is the origin of the radius sensitivity to α. Thus in constructing a solar model, it is convenient to tweak α to produce solar models with a radius that matches precisely the solar radius (Demarque & Percy 1964). The resulting value of α depends on the details of the model atmosphere (which determines the surface boundary condition) and on the local opacities (see, e.g., Guenther et al. 1992). In this scheme, a single free parameter, *the mixing length parameter* (α), *is used to compensate not only for the distortions introduced by the MLA in modeling convection, but also for the uncertainties in modeling the structure of the outer layers.* This is done simply by adjusting the structure of the SAL, via the mixing length parameter. It should, therefore, not be surprising if this approach does not model the SAL faithfully. With the advent of helioseismology, it is now possible to test directly the reliability of the MLA in modeling the solar SAL structure.

2.2. *The Approach of Canuto-Mazzitelli* (*CM*)

Canuto & Mazzitelli (1991, 1992) have attempted to generalize the MLA by taking into account the whole spectrum of convective wavelengths and, in this sense, the CM formalism is an improvement on the MLA. Grossman, Narayan, & Arnett (1993) introduced a similar moment approach to account for the spectrum of convective wavelengths but did not fit the parameters of their theory to numerical or experimental simulations of convection. The parameters of Canuto & Mazzitelli's description of convection are based on the results of laboratory experiments of incompressible convection extrapolated to stellar conditions. As in the MLA, CM describe radiation in the diffusion approximation. Using the laboratory analogy, CM argue in favor of a mixing length equal to the local distance to the surface convection boundary, although some of their calculations relate the mixing length to the pressure scale height (see also the calculations of Stothers & Chin 1995 which are based on the CM formulation). For the purpose of this paper, it suffices to point out that whatever choice of mixing length parameter is made, the CM formulation predicts a lower efficiency of convection in the outer parts of the SAL than the MLA (Canuto 1995). This translates into a sharper peak for the SAL than the MLA. As a result, CM solar models have been found to yield better agreement than the MLA with the observed solar *p*-mode frequencies (Paterno et al. 1993). A thorough discussion of the seismic properties of solar models constructed with various formulations of convection, including the CM treatment, will appear in a forthcoming paper.

2.3. *Balmforth's Nonlocal Mixing Length Theory*

Balmforth (1992a, 1992b, 1992c) has extended the nonlocal mixing length formulation of Gough (1977). His calculations indicate that perturbations to the frequencies owing to the effects of turbulence on the structure of the solar model are of the order of 10 μHz, whereas corrections to the adiabatic frequencies from nonadiabatic losses due to turbulence are of the order of 2 μHz.

2.4. *The Convective Flux Approximation*

The convective flux approximation (CFA) (Lydon, Fox, & Sofia 1992, 1993) is based on the numerical simulations of deep convection by Chan & Sofia (1989), from which it draws expressions for the convective flux. This represents the first attempt to incorporate the results of numerical simulations of convection into solar models. In the CFA, radiation is decoupled from convection and is treated in the same way as in the MLA, i.e., by the diffusion approximation. Because the Chan-Sofia simulation describes deep convection, the properties of convection had to be extrapolated outside the range of validity of the simulation, to the shallow layers of the solar model. The solar SAL in the CFA model is less peaked and located slightly deeper below the photosphere than the MLA model. This is at odds with the CM model and the shallow convection simulations presented here which produce more peaked SALs. The CFA model represents a significant step forward in describing the properties of deep efficient convection. But like the MLA, it fails in the shallow layers and the SAL where the coupling between radiative and convective transport must be taken into account.

2.5. *Numerical Simulations of Turbulent Flows*

In the shallow layers, the importance of averaging the opacity horizontally was discussed by Deupree (1979) within the MLA scheme. Deupree & Varner (1980) also made use of two-dimensional hydrodynamic simulations of stellar convection, in which the transfer of radiation is treated in the diffusion approximation, to parameterize the variation with depth of the mixing length at temperatures below 10^4 K. Their modifications of the MLA was applied to model atmospheres for Sun-like main-sequence stars by Lester, Lane, & Kurucz (1982). The Deupree-Varner modeling, however, does not extend deep enough to permit the determination of the mixing length in the deep adiabatic layers of the convection zone.

More recent model atmospheres, in which convection is described by a two-dimensional hydrodynamic simulation of radiation hydrodynamics of time-dependent convection have been described by Steffen (1993) and by Ludwig et al. (1996). The latter extend deep enough to permit an evaluation of the specific entropy in the adiabatic layers of the Sun and therefore to provide the first estimate of the effective mixing length from first principles.

Also recently, Fuhrmann, Axer, & Gehren (1993) and van't Veer-Menneret & Mégessier (1996) have constructed grids of model atmospheres for dwarf stars using the MLA to describe convection. These authors found that in order to find consistency between the effective temperature derived from the continuum with the effective temperature derived from the absorption lines, it is necessary to use a low value of α (about 0.5) in the shallow atmospheric layers. They also noted that this value of α differs from the larger value (about 2.0) needed in the interior model to derive the correct radius.

Sophisticated simulations of convection in the shallow layers have been carried out by Nordlund and his collaborators (Stein & Nordlund 1989; Dravins & Nordlund 1990; Nordlund & Stein 1996). The radiative transfer equations are solved using the opacity binning method, from the radiative surface, and extend into the convectively unstable layers of high superadiabaticity (SAL). As mentioned earlier, the simulations of Stein & Nordlund (1989) form the basis of the recent studies of the solar SAL by Monteiro et al. (1995) and Rosenthal et al. (1995).

Finally, Kim and his collaborators (Kim 1993; see also Kim et al. 1995, 1996a, 1996b) have carried out similar and complementary numerical simulations of shallow convection in the Sun, using a different numerical technique. These calculations, which apply to the transition from the inner part of the SAL into the deeper layers of the solar convection zone, also treat the coupling of radiative and convective transport and include realistic physics in the solar model. These fully compressible and three-dimensional simulations consider a radiation-coupled, nonmagnetic, gravitationally stratified medium using a realistic equation of state and opacities. But because radiation is treated in the diffusion approximation, the Kim et al. models become invalid in the optically thin outermost layers of the Sun. The Kim and the Nordlund models are in good qualitative agreement in the region of overlap.

In this paper, we present solar models constructed with a simple parameterization of convective efficiency in the outer layers based on Kim's simulations. We introduce a linearly varying mixing length parameter to mimic the convective transport efficiency in Kim's simulations. The structure of the models and their $l = 0$–100 *p*-mode spectra are compared. Monteiro et al. (1995) and Rosenthal et al. (1995) have taken a different approach, incorporating the actual structure from numerical simulations of Stein & Nordlund (1989) into their solar model calculation. These two different methods achieve the same goal, however, which is to compensate for the overestimated convective efficiency of the MLA.

3. HELIOSEISMOLOGY

Even though only the very high l ($l > 2000$) modes are confined to layers near and above the SAL of the Sun, all *p*-modes are to varying degrees affected by the structure of the SAL (Monteiro et al. 1995). In general, as the frequency of the *p*-modes increase, so do their sensitivity to perturbations to the structure (see, e.g., Christensen-Dalsgaard 1986). The integrated kinetic energy of an oscillation mode (often called the mode mass) decreases dramatically with increasing frequency and with increasing l. To first order, the perturbation to the frequency resulting from a perturbation to the structure is inversely proportional to the mode mass and as a consequence small structural changes lead to correspondingly larger frequency perturbations at higher frequencies and at higher l-values. To facilitate comparisons of the frequency perturbations of different l-values *at similar frequencies*, one can multiply each frequency difference (model frequency minus observed frequency) by the corresponding mode mass and divide by the mode mass of a radial mode at the same frequency. This eliminates the l dependence of the frequency perturbation. This weighting is important when considering a wide range of l-values, which we hope to be able to do when accurate high l p-mode data become available, but is less important, as we show in § 4.3, for the range of l-values considered here.

As described in the next section, we constructed several solar models with different variable-α curves and then calculated and compared their $l \leq 100$ p-mode frequencies to the observed frequencies (Libbrecht, Woodard, & Kaufman 1990). Although not as mathematically elegant as carrying out a formal inversion of the p-mode oscillations to determine the inferred structure (subject to the unknown uncertainties in the equation of state and surface boundary conditions), our forward approach does guarantee that only solutions that are physically realistic solutions of the (nonlinear) equations of stellar structure are considered. More importantly, it enables us to easily compare our variable-α curves, which quantify the efficiency of convection as a function of depth, with Kim's numerical simulations of convection.

4. RESULTS

4.1. *Variable Mixing Length Experiment*

From Kim's numerical simulations of convection we obtained the convective flux F_c and the mean convective velocity v_c from which we derived an effective mixing length parameter, α, using equation (7) of Kim et al. (1996a), i.e., $\alpha = (4/F_c)\,(C_p/QR)\,\rho\mu v_c^3$, where C_p is the specific heat at constant pressure, Q is the expansion coefficient of a convective element at constant pressure $\{Q = [(d\ln\rho)/(d\ln T)]p\}$, T is the temperature, R is the gas constant, ρ is the density, and μ is the mean molecular weight. The effective α is plotted opposite the natural logarithm of the pressure divided by the surface pressure ($p_{\rm top}$) in Figure 1. The effective mixing length parameter α does not remain constant throughout the region modeled by the shallow convection simulations, as would be prescribed by the MLA, but decreases as the surface is approached. Within a region close to the surface (one pressure scale height) the drop-off in α is due to the fixed and impenetrable boundary conditions imposed at the surface in Kim's numerical simulations. But, as discussed in Kim et al. (1996b), the drop-off in deeper regions cannot be attributed to the surface boundary conditions, and represents a real decrease in the contribution of convection to the total energy flux as the surface is approached. Numerical simulations are now being carried out by one of us (Y. C. K.), in collaboration with K. L. Chan, which include more realistic surface boundary conditions, that will enable us to simulate and resolve the SAL fully.

To mimic the drop-off in α in our solar model calculation we defined a variable mixing length parameter α by

$$\alpha = a_1 \ln(p/p_{\rm top}) + a_2\ ,$$

where $p_{\rm top} \equiv 1.3 \times 10^5$ dyn cm^{-2} is approximately the pressure near the surface of our solar model. Below the superadiabatic peak we held α constant once it reached a maximum value of 8. This has no affect on the structure because the temperature gradient is nearly the adiabatic gradient below the superadiabatic peak. We constructed models for $a_1 = 0.5$, 1.0, and 1.5. The value of a_2 was adjusted to produce solar models with the observed radius. Figure 1 shows the resultant variation in α as a function of pressure for the three solar models corresponding to $a_1 = 0.5$, 1.0, and 1.5. The solar models are identical in all other respects to the *opal + diff* model in Guenther, Kim, & Demargue (1996). The OPAL equation of state and opacities were used. Helium diffusion was included using the Bahcall & Loeb (1990) description.

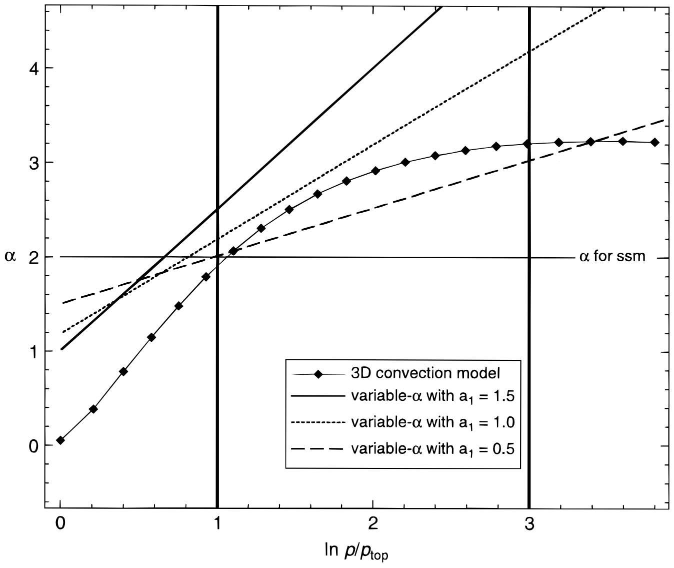

FIG. 1.—The implied value of the mixing length parameter from numerical simulations of shallow convection is plotted opposite the natural logarithm of the pressure, scaled to the surface pressure $p_{\rm top} = 1.3 \times 10^5$ dyn cm^{-2}. To simulate the variable α in our solar models we allowed α to vary linearly as shown by the three straight lines. The lines have slopes 0.5, 1.0, and 1.5. The offset, or surface intercept of the lines, was forced by the condition that the solar model have the observed radius. Owing to the crude boundary conditions applied at the surface and the base in our numerical simulation of convection, we ignore the simulation results in the region within one pressure scale height of the surface of the simulation and the region within one pressure scale height of the base of the simulation (indicated by thick vertical lines). The horizontal line corresponds to the constant α used in the standard solar model (ssm).

4.2. *Superadiabatic Peak*

As with the CM formalism, our variable mixing length parameter models have higher superadiabatic peaks than solar models that use the fixed α of the MLA (see Fig. 2). As we increase a_1, i.e., the slope in the α versus $\ln(p/p_{\rm top})$ relationship, the superadiabatic peak increases. In this manner, our variable-α model permits us to control the efficiency of convection: increasing a_1 increases the superadiabatic peak which corresponds to a decrease in convective efficiency. Note that because all the models are calibrated to the solar radius, they are characterized by the same specific entropy in the convection zone, and have the same interior structure.

For all three variable-α models, the surface intercept for α is nonzero. This is a consequence of the fact that the adiabat of the convection zone is determined by the value of α inward of the superadiabatic peak; and the radius of the model depends on this adiabat (temperature gradient). When the slope of the variable-α relationship is fixed, the only way to adjust α in this region, thereby adjusting the adiabat, and consequently the radius, is by raising or lowering the surface intercept. Figure 1 shows that, unlike the numerical simulation where v_c (and therefore also α) is constrained to reach zero at the surface, the finite value of the

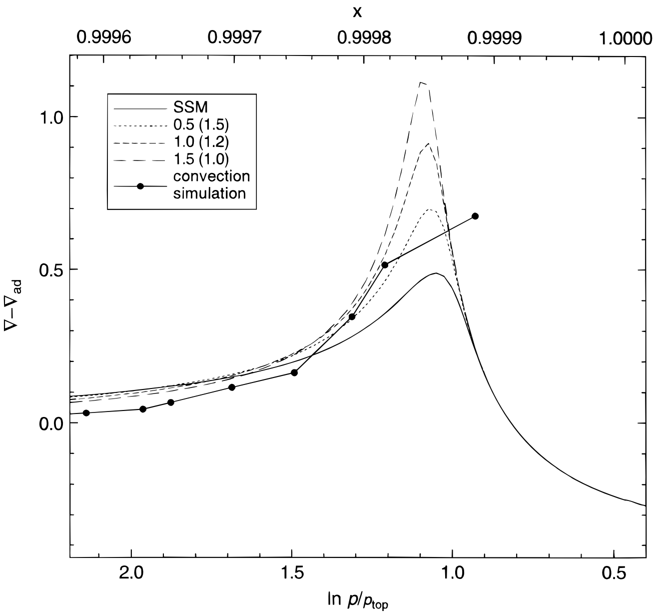

FIG. 2.—The linearly varying mixing length parameter models (see Fig. 1) alter the structure of the outer layers. Here we show the superadiabatic gradient plotted opposite the radius fraction (scale along top) and the scaled pressure (scale along bottom) for the standard solar model with a constant mixing length parameter, and the three linearly varying mixing length parameter models. The increased superadiabatic peak for the variable α models corresponds to less efficient convection. For illustration, the run of the horizontally averaged superadiabatic gradient from the Kim et al. (1995) simulation is also shown. We remind the reader that for a number of reasons discussed in the text, the simulation and the solar models are not strictly comparable, but we note the remarkable coincidence in trend between the convection simulation and the variable-α models.

intercept a_2 in our formula implies that α does not vanish at the surface. Although this behavior is suggestive of the presence of convective overshoot at the top boundary (convective overshoot may play a role in the Sun), we cannot attach any particular physical significance to this parameter. Like the conventional MLA, in which α does not vanish at the convection boundary, our modified MLA formalism contains no description of convective overshoot.

We stress that the linear variation in α is suggested only by the drop-off in α in the numerical simulations of Kim and is not based on any other improvement to the MLA itself regarding the detailed properties of convective flows. Our goal is to show that by mimicking, in the crudest manner, the behavior implied by our preliminary numerical simulations of convection, we can improve the structure of the model in the SAL. This is illustrated in the next section.

4.3. *Helioseismology*

Comparing the frequency difference plots Figure 3*a* (model minus solar observations) for the standard MLA model and for three variable-α models, one sees immediately the benefit of the variable mixing length. For the best solar model, which is based on the latest physics (including helium diffusion and the OPAL equation of state and opacities) and uses the standard MLA, the discrepancy in *p*-mode frequencies increases with increasing frequency. This discrepancy is almost eliminated in the variable-α models, in which α decreases as the surface is approached.

Figure 3*b* is similar to Figure 3*a* except that the frequency differences are weighted by the factor $Q_{n,l}(\nu_{n,l})$ which removes the *l* dependence of the sensitivity of the frequency discrepancies. $Q_{n,l}$ is defined in Christensen-Dalsgaard (1986) and is essentially equal to the integrated kinetic energy of the *p*-mode, $\nu_{n,l}$, divided by the integrated kinetic energy of a radial mode ($l = 0$) of similar frequency. The thickness of the bundle of lines is reduced by a factor of 2. This is expected (see discussion in § 3). The $l = 100$ *p*-modes will be perturbed twice as much for a given structural perturbation as the $l = 0$ *p*-modes because the integrated kinetic energy of the $l = 100$ *p*-modes are approximately one-half that of the $l = 0$ *p*-modes (at similar frequency). The weighting, therefore, equalizes their response to a given structural perturbation.

The spread in the bundle of lines indicates that structural discrepancies remain between the model and the Sun. But we see that the variable-α models do reduce the discrepancy in *p*-mode frequencies, especially the progressive increase in discrepancy with increasing frequency (slope error), which is caused by errors in the structure of the very outermost layers of the model (Guenther et al. 1996). This is seen directly by comparing the sound speed profiles of the models.

The frequencies of the *p*-modes depend on the run of sound speed in the model, which in turn depend on the density profile and the run of Γ_1. In Figure 4 we show the effects of our variable-α modeling on the sound speed by plotting the sound speed of the variable-α models minus the sound speed of the standard solar model (SSM). As was suggested by the reduction in the slope error, we see that the sound speed in the outermost layers is perturbed by the variable-α modeling only in the region immediately surrounding the superadiabatic peak. From Figure 5 we see that the adiabatic gradient Γ_1 is also perturbed by the variable-α modeling only in the vicinity of the superadiabatic peak. The small perturbation to the sound speed near $x = 0.7$ seen in Figure 4 is an artifact of the slightly different amounts of diffuse helium that exists immediately below the convection zone base.

Inspection of Figure 1 and Figure 3 further reveal that of the three variable-α models, the one on which α is closest to the value of α in the standard MLA model, particularly in the region of the superadiabatic peak (in this case, as in the equivalent model of Guenther et al. 1996, α is near 2), i.e., the model for which $a_1 = 0.5$, is also the one that yields the best agreement with the observed solar frequencies. Also the conclusion that the effective α is nearly constant at greater depth seems consistent with the observation by Chan & Sofia (1987, 1989), based on their numerical simulations, that in the limit of deep convection (and below the ionization zones) there exists a correlation length which is directly proportional to the local pressure scale height.

We wish to reemphasize that we have carefully avoided using the precise α-profile from the horizontally averaged superadiabatic gradient in the simulation of Kim et al. (1996a) shown in Figure 1. It would have been incorrect for several reasons. Even though the initial conditions in the convection simulation were taken from a standard solar model and the simulation itself included realistic local physics, several approximations had been made, e.g., in the use of the diffusion approximation to describe radiative transfer, and in the rigid boundary at the surface. Most importantly, the α-profile from the simulation was not subject to the usual constraint of solar models, that of matching the radius of the model to the solar radius. Further, the precise choice of α in a standard solar model is

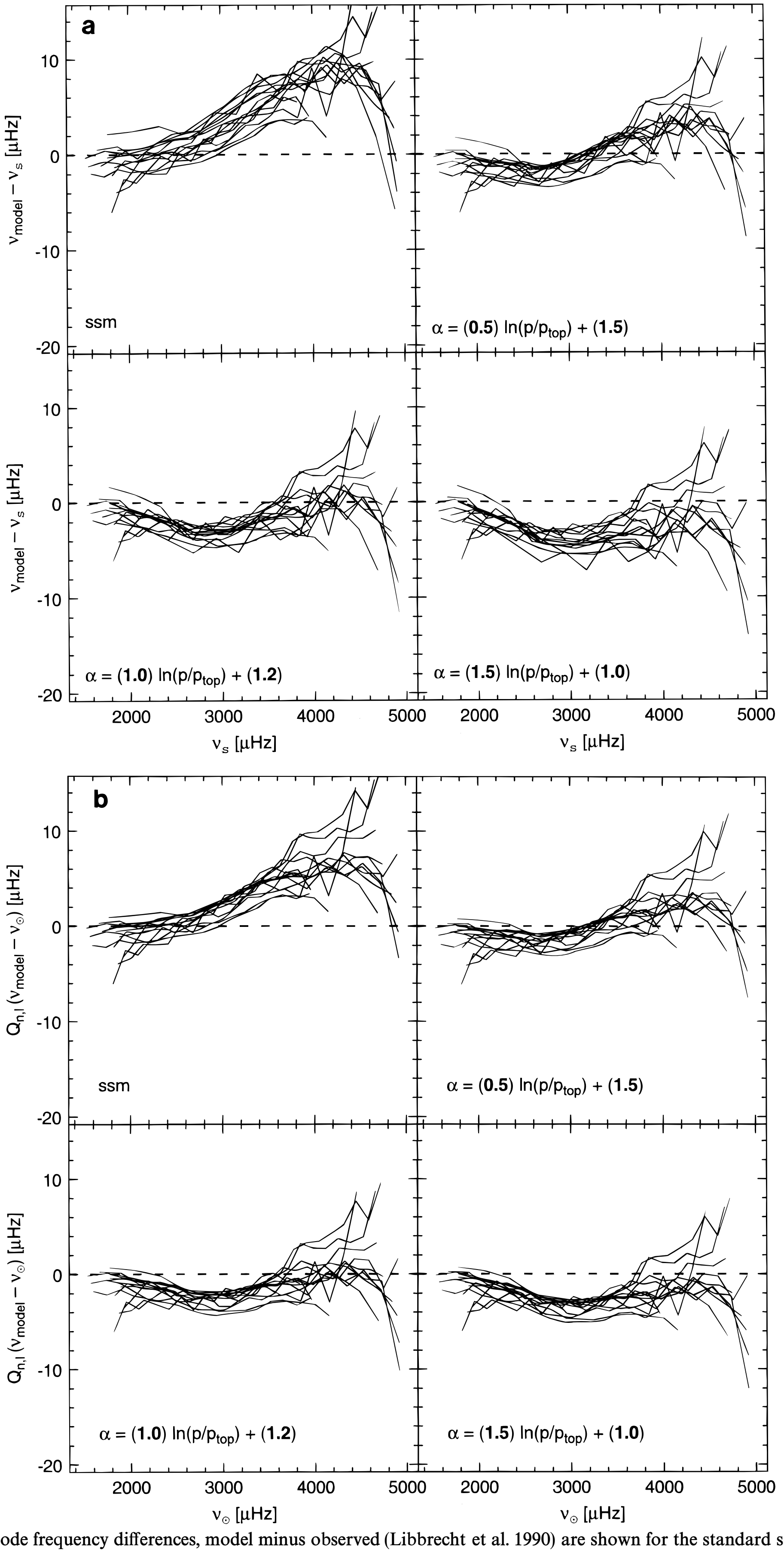

FIG. 3.—(*a*) The *p*-mode frequency differences, model minus observed (Libbrecht et al. 1990) are shown for the standard solar model (ssm) and the three variable α solar models. All observed *p*-modes with $l = 0, 1, 2, 3, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100$ are plotted with lines joining common *l*-values. (*b*) Similar to Fig. 3*a*, except the frequency differences are weighted by the factor $Q_{n,l}$ which is the integrated kinetic energy of the mode divided by the integrated kinetic energy of an $l = 0$ mode with the same frequency. This weighting scales out the different sensitivities of the frequency perturbations to *l*.

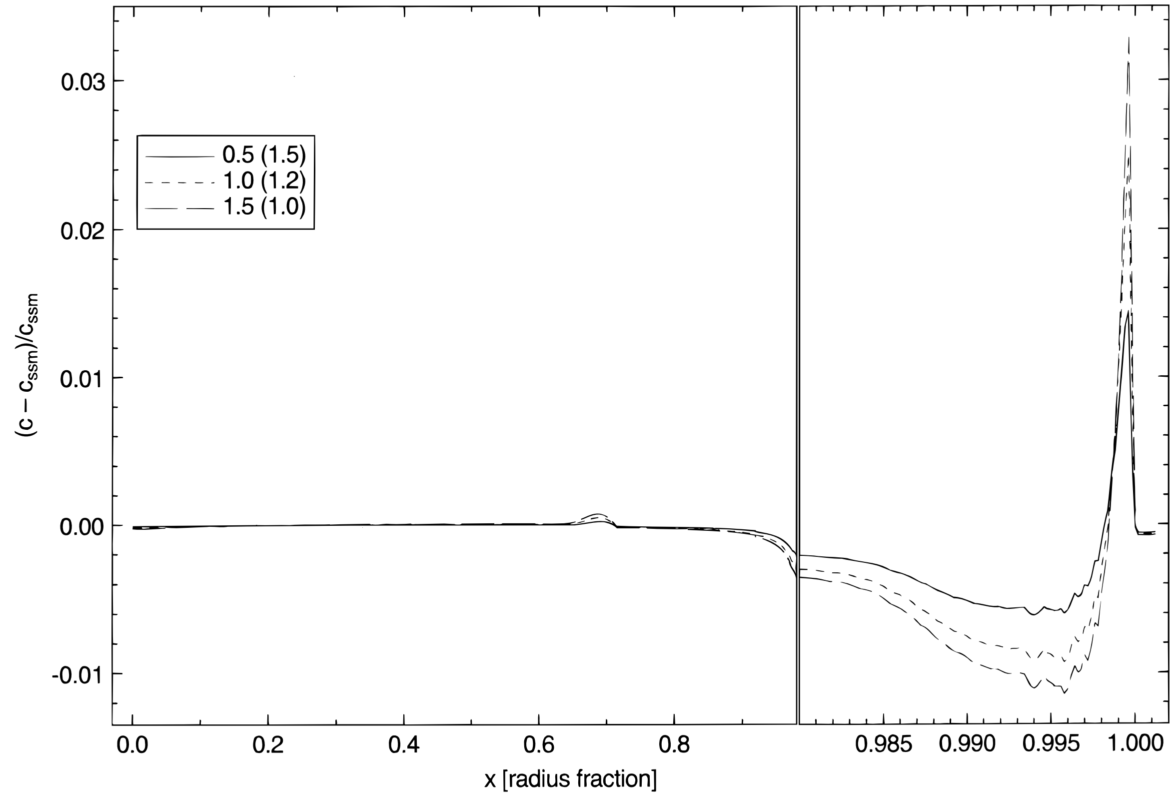

FIG. 4.—The run of the sound speed differences between the variable-α models and the standard solar model (ssm) are plotted as a function of radius fraction. The x-axis scale is enlarged near the surface in the region of the superadiabatic peak.

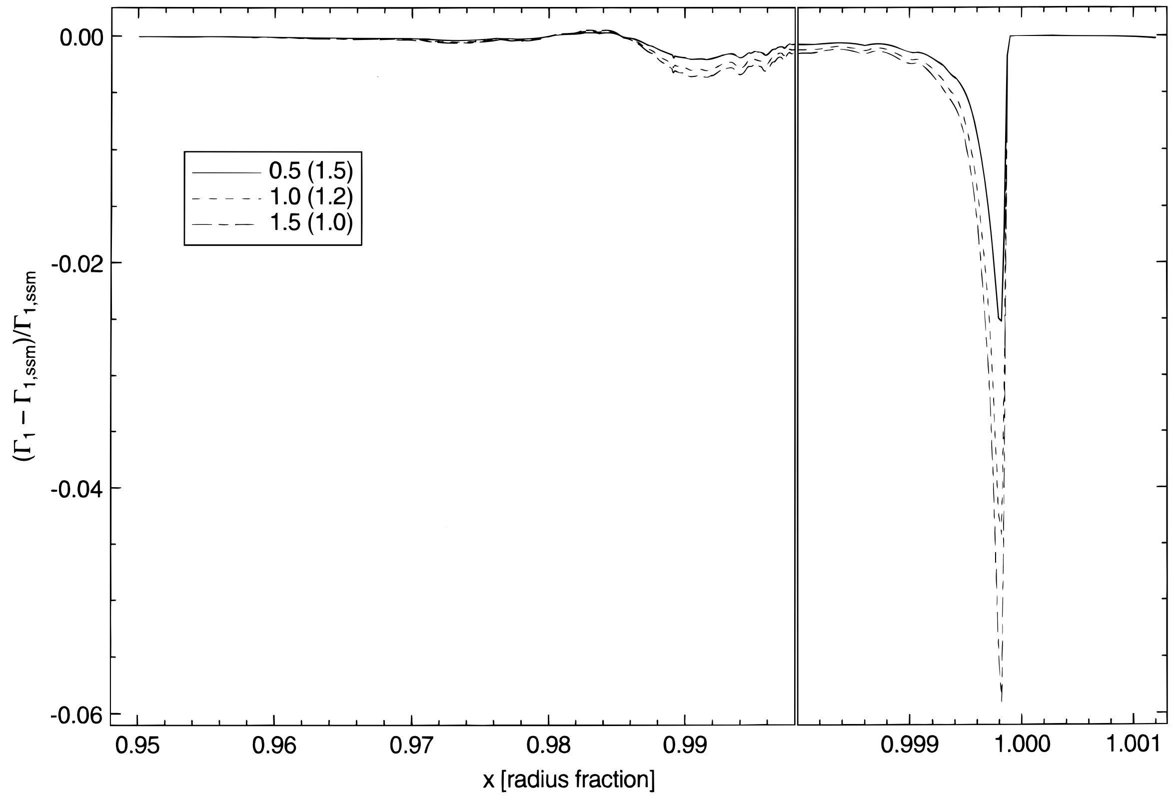

FIG. 5.—The run of the adiabatic exponent, Γ_1 differences between the variable-α models and the standard solar model (ssm) are plotted as a function of radius fraction. The x-axis scale is enlarged near the surface in the region of the superadiabatic peak.

a function of the opacities chosen, and the Kim et al. (1995) calculations were based on the Los Alamos Opacity Library (Huebner et al. 1977), rather than the Livermore OPAL opacities used here, which are substantially larger at low temperatures. These differences should be kept in mind as well when comparing the run of superadiabaticity in the Kim et al. (1995) simulation with the models of this study, as is done in Figure 2.

The α-profile from the Kim et al. (1995) simulation can therefore be viewed only as a guide. In addition, we do not believe, or wish to suggest, that the errors in the calculated high-frequency modes are entirely due to the treatment of the convective layers. The uncertainties in the low-temperature opacities and in the surface boundary conditions (the atmosphere model) also perturb the p-modes at the higher frequencies at the level shown here (Guenther et al. 1992). For all these reasons, we must first improve our modeling of radiation and the sophistication of the surface boundary condition in our numerical simulations of convection before deducing the actual efficiency of convection near the surface of the Sun.

5. CONCLUSIONS

Guided by numerical simulations of shallow convection in the Sun, which show that the conventional MLA underestimates the ratio of radiative to convective transport in the outer layers of the solar convection zone, we have investigated models in which the mixing length parameter α increases with depth in the SAL. We have found that, by comparing the p-mode frequencies of the models with the observed p-mode frequencies of the Sun, the variable-α models are in better agreement with the Sun than models based on conventional MLA. This result is consistent with the conclusion of Rosenthal et al. (1995), who attached a surface layer structure derived from the Stein & Nordlund (1989) solar atmospheric simulations and found similar improvements to the p-mode frequencies.

We will prepare a more detailed *forward inversion* of the solar SAL, which will include comparisons to Canuto & Mazzitelli's formulation, when we receive high-l p-mode frequency data. With the $l \leq 100$ p-modes we are currently using, the different formulations of convection are not distinguishable.

Ultimately we hope to be able to model the convective envelope in the Sun and other stars from first principles sufficiently well to calculate reliable stellar radii, free from the uncertainties of the mixing length formalism. We are presently improving our numerical simulations and expect to develop increasingly more realistic convection simulations including a consistent treatment of the radiative-convective transition layer within the next year. We are concerned about the possible variation in convective efficiency for stars in different phases of evolution. Here, it is hoped, seismic observations of other stars will offer enough information to constrain the convection models. Indeed, some initial progress is already being made (Monteiro et al. 1995; Guenther & Demarque 1996; Christensen-Dalsgaard et al. 1995) using recent p-mode observations of the subgiant η Boo by Kjeldsen et al. (1995).

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