Double-ended queues and joint moments of left-right canonical operators on full Fock space

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Abstract

We follow the guiding line offered by canonical operators on the full Fock space, in order to identify what kind of cumulant functionals should be considered for the concept of bi-free independence introduced in the recent work of Voiculescu. By following this guiding line we arrive to consider, for a general noncommutative probability space $(\mathcal{A}, \varphi)$, a family of "(ℓ, r)-cumulant functionals" which enlarges the family of free cumulant functionals of the space. In the motivating case of canonical operators on the full Fock space we find a simple formula for a relevant family of (ℓ, r)-cumulants of a $(2d)$-tuple $(A_1, \ldots, A_d, B_1, \ldots, B_d)$, with $A_1, \ldots, A_d$ canonical operators on the left and $B_1, \ldots, B_d$ canonical operators on the right. This extends a known one-sided formula for free cumulants of $A_1, \ldots, A_d$, which establishes a basic operator model for the R-transform of free probability.

Keywords: bi-free probability, canonical operators, double-ended queues, bi-non-crossing partitions, bi-free cumulants.
Mathematics Subject Classification 2000: Primary 46L54; Secondary 68R05.

1. Introduction

In this paper [2] we follow the guiding line offered by a special type of canonical operators on the full Fock space, in order to identify what kind of cumulant functionals should be considered for the concept of *bi-free independence* introduced by D. Voiculescu [8], [9].

In Section 1.1 we give a general (rather informal) description of what is the above mentioned "guiding line"; the main point of the description is that we are upgrading from (I) to (II) in the diagrams displayed on the next page.

1.1 From (bi-)free independence to partitions, via canonical operators.

The concept of free independence for noncommutative random variables was pinned down by D. Voiculescu in the 1980's ([6], [7]), with inspiration from natural families of generators for algebras associated to free groups. An important further idea brought by R. Speicher [5] was that calculations with freely independent random variables can be efficiently done by using the lattices $NC(n)$ of non-crossing partitions, and free cumulant functionals based on these lattices. The main point of the cumulant approach is that free independence is equivalent to a condition (many times easier to verify than the original definition) of "vanishing of mixed free cumulants" for the random variables in question.

What is the guiding line which allows one to start from the original definition of free independence, based on free groups, and "find" the $NC(n)$'s? Here we pursue (with due

[1]Research supported by a Discovery Grant from NSERC, Canada.
[2]This is the electronic version of an article published in International Journal of Mathematics 26 (2015), Issue 02, paper 1550016. DOI: 10.1142/S0129167X15500160,

acknowledgement that there is more than one answer to the question just asked) the line provided by a special type of "canonical" operators on the full Fock space $\mathcal{T}_d$ over $\mathbb{C}^d$. These are certain power series in creation/annihilation operators on $\mathcal{T}_d$, which were invented in [7] for $d = 1$, then extended in [3] to general d. The fact that they play a role in free probability is not so surprising, since $\mathcal{T}_d$ itself is in a certain sense an incarnation of the free monoid on d generators. But now, the way how products of canonical operators act on the vacuum-vector of $\mathcal{T}_d$ is encoded by the action of inserting and removing objects in a well-known gadget from theoretical computer science, called *lifo-stack* (lifo = abbreviation for "last-in-first-out"). Lifo-stacks are closely connected (through the intermediate of some lattice paths called Lukasiewicz paths) to non-crossing partitions, so overall we get a diagram like this:

$$\text{(I)} \quad \begin{pmatrix} \text{free independence for} \\ d \text{ random variables} \end{pmatrix} \longrightarrow \begin{pmatrix} d\text{-tuple of canonical} \\ \text{operators on full Fock space} \end{pmatrix}$$
$$\longrightarrow \begin{pmatrix} \text{lifo-} \\ \text{stacks} \end{pmatrix} \longrightarrow \begin{pmatrix} \text{non-crossing} \\ \text{partitions} \end{pmatrix}.$$

In 2013, Voiculescu [8], [9] started to study the concept of *bi-free independence* for d *pairs* of random variables in a noncommutative probability space. This too is a concept inspired from looking at algebras associated to free groups, but its definition is in some sense indirect – it ultimately boils down to a special way of representing the $2d$ random variables in question on a free-product space. In order to advance the study of bi-free independence, it is of obvious importance (even more so than it was the case for plain free independence) to be able to re-phrase it as a vanishing condition on mixed cumulants, for a suitable construction of cumulant functionals. The candidate for how to construct such "bi-free cumulant functionals" will come out of an upgrade of diagram (I), which tells us what lattices of partitions to use in the stead of $NC(n)$'s. The upgrade is as follows:

$$\text{(II)} \begin{pmatrix} \text{bi-free independence for} \\ d \text{ pairs of random variables} \end{pmatrix} \xrightarrow{(1)} \begin{pmatrix} (2d)\text{-tuple of canonical} \\ \text{operators on full Fock space,} \\ d \text{ on left and } d \text{ on right} \end{pmatrix}$$
$$\xrightarrow{(2)} \begin{pmatrix} \text{double-ended} \\ \text{queues} \end{pmatrix} \xrightarrow{(3)} \begin{pmatrix} \text{bi-non-crossing} \\ \text{partitions} \end{pmatrix}.$$

(We have numbered the three arrows which appear in diagram (II), in order to discuss them separately in the next subsection.)

The lattices of partitions obtained in (II) are indexed not only by a positive integer n, but also by a tuple $\chi = (h_1, \ldots, h_n)$ where every h_i is either the letter ℓ (for "left") or the letter r (for "right"). Throughout the paper, the notation used for such a lattice of partitions will be $\mathcal{P}^{(\chi)}(n)$. For every $n \in \mathbb{N}$ and $\chi \in \{\ell, r\}^n$ we have that $\mathcal{P}^{(\chi)}(n)$ is a collection of partitions of $\{1, \ldots, n\}$. If $\chi = (\ell, \ell, \ldots, \ell)$ or if $\chi = (r, r, \ldots, r)$, then $\mathcal{P}^{(\chi)}(n) = NC(n)$, but for arbitrary $\chi \in \{\ell, r\}^n$ we generally have $\mathcal{P}^{(\chi)}(n) \neq NC(n)$.

1.2 Discussion of the three arrows in diagram (II).

Discussion of the connection "$\xrightarrow{(1)}$". Let d be a fixed positive integer, and let us also fix an orthonormal basis $e_1, \ldots, e_d$ for $\mathbb{C}^d$. Recall that the full Fock space over $\mathbb{C}^d$ is

$$\mathcal{T}_d = \mathbb{C} \oplus \mathbb{C}^d \oplus (\mathbb{C}^d \otimes \mathbb{C}^d) \oplus \cdots \oplus (\mathbb{C}^d)^{\otimes n} \oplus \cdots \tag{1.1}$$

In $\mathcal{T}_d$ we have a preferred orthonormal basis, namely

$$\{\xi_{\text{vac}}\} \cup \{e_{i_1} \otimes \cdots \otimes e_{i_n} \mid n \geq 1,\, 1 \leq i_1, \ldots, i_n \leq d\}, \tag{1.2}$$

where ξ_{vac} is a fixed unit vector in the first summand $\mathbb{C}$ on the right-hand side of (1.1). For every $1 \leq i \leq d$ we denote by $L_i, R_i \in B(\mathcal{T}_d)$ the left-creation and respectively the right-creation operator on $\mathcal{T}_d$ defined by the vector e_i. These are isometries which act on the preferred basis by $L_i(\xi_{\text{vac}}) = R_i(\xi_{\text{vac}}) = e_i$ and by

$$L_i(e_{i_1} \otimes \cdots \otimes e_{i_n}) = e_i \otimes e_{i_1} \otimes \cdots \otimes e_{i_n}, \quad R_i(e_{i_1} \otimes \cdots \otimes e_{i_n}) = e_{i_1} \otimes \cdots \otimes e_{i_n} \otimes e_i,$$

for $n \geq 1$ and $1 \leq i_1, \ldots, i_n \leq d$.

Suppose we are given a polynomial f without constant term in non-commuting indeterminates $z_1, \ldots, z_d$. Write it explicitly as

$$f(z_1, \ldots, z_d) = \sum_{n=1}^{\infty} \sum_{i_1, \ldots, i_n = 1}^{d} \alpha_{(i_1, \ldots, i_n)} z_{i_1} \cdots z_{i_n}, \tag{1.3}$$

where the α's are in $\mathbb{C}$ (and $\exists\, n_o \in \mathbb{N}$ such that $\alpha_{(i_1, \ldots, i_n)} = 0$ for $n > n_o$). For $1 \leq i \leq d$, let A_i be the operator in $B(\mathcal{T}_d)$ defined as follows:

$$A_i := L_i^* \Big(I + \sum_{n=1}^{\infty} \sum_{i_1, \ldots, i_n = 1}^{d} \alpha_{(i_1, \ldots, i_n)} L_{i_n} \cdots L_{i_1} \Big). \tag{1.4}$$

We will say that $(A_1, \ldots, A_d)$ is the *d-tuple of left canonical operators* with symbol f. The reason to care about this d-tuple is that it provides one of the possible approaches (historically the first) to the R-transform, a fundamental tool used in free probability. More precisely: if we endow $B(\mathcal{T}_d)$ with the vacuum-state φ_{vac} (i.e. with the vector-state defined by the vector ξ_{vac} from (1.2)), then the R-transform of $(A_1, \ldots, A_d)$ with respect to φ_{vac} is [3] the given f. For a detailed presentation of how this goes, see Lecture 21 of [4]. Let's also note that, as explained in that lecture of [4] (Theorem 21.4 there), the derivation of the R-transform of $(A_1, \ldots, A_d)$ relies solely on the fact that $L_1, \ldots, L_d$ used in Equation (1.4) form a free family of Cuntz isometries. The latter fact means, by definition, that the L_i's are isometries with $L_i^* L_j = 0$ for $i \neq j$, and that one has

$$\varphi_{\text{vac}}(L_{i_1} \cdots L_{i_m} L_{j_1}^* \cdots L_{j_n}^*) = 0$$

for all non-negative integers m, n with $m + n \geq 1$ and all $i_1, \ldots, i_m, j_1, \ldots, j_n \in \{1, \ldots, d\}$.

Since the creation operators $R_1, \ldots, R_d$ on the right also form a free family of Cuntz isometries, it follows that canonical d-tuples of operators could be equivalently constructed by using creation and annihilation operators *on the right side* (instead of the left side favoured in Equation (1.4)). For the present paper it is important to also write explicitly this second set of formulas. So let g be a polynomial without constant term in the same $z_1, \ldots, z_d$, and let us write it explicitly as

$$g(z_1, \ldots, z_d) = \sum_{n=1}^{\infty} \sum_{i_1, \ldots, i_n = 1}^{d} \beta_{(i_1, \ldots, i_n)} z_{i_1} \cdots z_{i_n} \tag{1.5}$$

[3] So what we have here is a canonical construction which produces a d-tuple of operators with prescribed R-transform. In this discussion f could be a formal power series in $z_1, \ldots, z_d$, in which case $A_1, \ldots, A_d$ would live in a suitable algebra of formal operators on $\mathcal{T}_d$, as described for instance on pp. 344-346 of [4]. For the sake of not complicating the notations, we will stick to f being a polynomial.

(with β's in $\mathbb{C}$, and where $\exists\, n_o$ such that $\beta_{(i_1,\ldots,i_n)} = 0$ for $n > n_o$). For $1 \le i \le d$, we put

$$B_i := R_i^* \Big(I + \sum_{n=1}^{\infty} \sum_{i_1,\ldots,i_n=1}^{d} \beta_{(i_1,\ldots,i_n)} R_{i_n} \cdots R_{i_1} \Big) \in B(\mathcal{T}_d). \tag{1.6}$$

Then $(B_1, \ldots, B_d)$ is called the *d-tuple of right canonical operators* with symbol g, and has the property that its R-transform with respect to φ_{vac} is the polynomial g we started with.

If taken in isolation, the B_i's from Equation (1.6) would merely duplicate what the A_i's from (1.4) are already doing. What is interesting is to consider the *combined $(2d)$-tuple* of A_i's and B_i's – this provides a significant example of d pairs of left/right variables, as one wants to study in bi-free probability.

Discussion of the connection "$\xrightarrow{(2)}$". We now examine the values of φ_{vac} on monomials made with the canonical operators $A_1, \ldots, A_d, B_1, \ldots, B_d$ that were considered above. Let us first recap the one-sided case, where we look at an expectation

$$\varphi_{\mathrm{vac}}(A_{j_1} \cdots A_{j_n}) = \langle A_{j_1} \cdots A_{j_n} \xi_{\mathrm{vac}} \, , \, \xi_{\mathrm{vac}} \rangle,$$

for some $n \ge 1$ and $1 \le j_1, \ldots, j_n \le d$. If we replace each of $A_{j_1}, \ldots, A_{j_n}$ by using (1.4), and if we expand the ensuing product of sums, then we arrive to act on ξ_{vac} with products of the form

$$L_{j_1}^* (L_{i_{1,1}} \cdots L_{i_{1,m(1)}}) \cdots L_{j_n}^* (L_{i_{n,1}} \cdots L_{i_{n,m(n)}}), \tag{1.7}$$

where $m(1), \ldots, m(n) \ge 0$ are such that $m(1) + \cdots + m(n) = n$. The action of such a product on ξ_{vac} can be followed intuitively by thinking of how a collection of n balls ①, …, ⓝ moves through a lifo-stack – the balls go into the stack in [4] groups (creation) and are taken out of the stack one by one (annihilation).

What happens when we upgrade to a monomial which contains both some A_i's and some B_i's? We can still proceed in the same way as above, only that in (1.7) some of the L_i's and L_j^*'s become R_i's and R_j^*'s. The intuition of n balls moving through a device like a stack continues to work, but sometimes (when we have "R" instead of "L") the balls must go in or out "by the other end of the stack". This is precisely the device which in theoretical computer science goes under the name of *double-ended queue*, or *deque* for short – see e.g. Section 2.2 of [2]. Some pictures showing how we think about deques and how we use them in the present paper appear on pages 11-12 in Section 3.

Discussion of the connection "$\xrightarrow{(3)}$". The set of partitions $\mathcal{P}^{(\chi)}(n)$. Once again, let us first recap the one-sided case, where we look at n balls moving through a lifo-stack. Every possible scenario of how the balls move through the stack has associated to it a certain partition of $\{1, \ldots, n\}$, which we call "output-time partition" (see Definition 3.3, Example 3.4.2 below). When we pursue the discussion started in (1.7), the concrete formula obtained for $\varphi_{\mathrm{vac}}(A_{j_1} \cdots A_{j_n})$ comes out as a sum indexed by all possible output-time partitions. But the last-in-first-out rule of the stack forbids output-time partitions from having crossings! What has come out is precisely a summation formula over $NC(n)$ (as we knew it should).

What happens when we upgrade to the case of combined A_i's and B_i's? We now have n balls moving through a deque. We still have the concept of output-time partition associated to a scenario for how the balls move through the deque, but this partition may now have

[4] The sizes of the groups of balls going in the stack are $m(n), \ldots, m(2), m(1)$. Here we ignore for the moment some relations that must also be imposed in between the indices $i_{1,1}, \ldots, i_{n,m(n)}$ and $j_1, \ldots, j_n$.

crossings, due to the interference between left and right moves (e.g. it is possible that some of the balls ①, . . . , ⓝ enter the deque by one side and exit by the other).

If we fix a tuple $\chi \in \{\ell, r\}^n$, then the set $\mathcal{P}^{(\chi)}(n)$ of bi-non-crossing partitions corresponding to χ will be defined in Section 3 of the paper as the set of all partitions of $\{1, \ldots, n\}$ which can arise as output-time partition for a deque-scenario compatible with χ. (See details in Definition 3.5.) In the case when χ happens to be either $(\ell, \ldots, \ell)$ or $(r, \ldots, r)$, then the deque is reduced to a lifo-stack, and $\mathcal{P}^{(\chi)}(n)$ is equal to $NC(n)$. For general $\chi \in \{\ell, r\}^n$, it follows immediately from the definition that $\mathcal{P}^{(\chi)}(n)$ has the same cardinality as $NC(n)$, but $\mathcal{P}^{(\chi)}(n)$ is generally different from $NC(n)$ itself.

In Section 4 of the paper we will find (Theorem 4.10) an alternative description for $\mathcal{P}^{(\chi)}(n)$. It says that $\mathcal{P}^{(\chi)}(n) = \{\sigma_\chi \cdot \pi \mid \pi \in NC(n)\}$, where σ_χ is a specific (concretely described) permutation of $\{1, \ldots, n\}$ associated to χ, and where the action of a permutation σ on a partition π is defined in the natural way (if $\pi = \{V_1, \ldots, V_k\}$, then $\sigma \cdot \pi = \{\sigma(V_1), \ldots, \sigma(V_k)\}$). This alternative description of $\mathcal{P}^{(\chi)}(n)$ is very useful for concrete calculations. It also gives immediately the fact that, with respect to reverse refinement order, $\mathcal{P}^{(\chi)}(n)$ is a lattice isomorphic to $NC(n)$.

1.3 From partitions to cumulant functionals.

In both classical and free probability theory, the standard method to introduce cumulant functionals goes by writing a so-called "moment-cumulant" formula, where moments are expressed in terms of cumulants via summations over a suitable family of lattices of partitions. In particular, free cumulant functionals (as introduced by Speicher [5]) have a moment-cumulant formula based on non-crossing partitions. To be specific, let a noncommutative probability space $(\mathcal{A}, \varphi)$ be given. The free cumulant functionals of $(\mathcal{A}, \varphi)$ are defined as the family of multilinear functionals $(\kappa_n : \mathcal{A}^n \to \mathbb{C})_{n=1}^\infty$ which is uniquely determined by the requirement that for every $n \geq 1$ and $a_1, \ldots, a_n \in \mathcal{A}$ we have:

$$\text{(M-FC)} \qquad \varphi(a_1 \cdots a_n) = \sum_{\pi \in NC(n)} \Big(\prod_{V \in \pi} \kappa_{|V|}(\,(a_1, \ldots, a_n) \mid V\,) \Big).$$

By the same token, the families of partitions $\mathcal{P}^{(\chi)}(n)$ discussed at the end of section 1.2 can be used to define a concept of cumulant functionals, as follows. Let a noncommutative probability space $(\mathcal{A}, \varphi)$ be given. There exists a family of multilinear functionals

$$\Big(\kappa_\chi : \mathcal{A}^n \to \mathbb{C} \Big)_{n \geq 1,\, \chi \in \{\ell, r\}^n}$$

which is uniquely determined by the requirement that for every $n \geq 1$, $\chi \in \{\ell, r\}^n$ and $a_1, \ldots, a_n \in \mathcal{A}$ we have:

$$\text{(M-F}_2\text{C)} \qquad \varphi(a_1 \cdots a_n) = \sum_{\pi \in \mathcal{P}^{(\chi)}(n)} \Big(\prod_{V \in \pi} \kappa_{\chi|V}(\,(a_1, \ldots, a_n) \mid V\,) \Big).$$

The explanation of various notational details, the easy proof of existence/uniqueness, and a bit of further discussion around the functionals κ_χ are given in Section 5 of the paper. We will refer to [5] these functionals as the (ℓ, r)-*cumulant functionals* associated to the

[5] In view of the results obtained in [1], after the present paper was first circulated, it is justified to also refer to the κ_χ's as "bi-free cumulant functionals" associated to $(\mathcal{A}, \varphi)$.

noncommutative probability space $(\mathcal{A}, \varphi)$. It is immediate that they provide an enlargement of the family of free cumulants $(\kappa_n)_{n=1}^{\infty}$ associated to the same space, in the respect that

$$\kappa_n = \kappa_{(\underbrace{\ell, \ldots, \ell}_{n})} = \kappa_{(\underbrace{r, \ldots, r}_{n})}, \quad n \geq 1.$$

1.4 Back to canonical operators: main result of the paper.

The concept of (ℓ, r)-cumulants was introduced in section 1.3 for a general noncommutative probability space $(\mathcal{A}, \varphi)$. Here we go back to the special space $(B(\mathcal{T}_d), \varphi_{\mathrm{vac}})$ from section 1.2. Let $(A_1, \ldots A_d)$ and $(B_1, \ldots B_d)$ be d-tuples of left (respectively right) canonical operators with symbols f and g, as in Equations (1.4) and (1.6). We are interested in the (ℓ, r)-cumulants of the combined $(2d)$-tuple $(A_1, \ldots, A_d, B_1, \ldots, B_d)$. The point we want to establish is that a relevant family of such (ℓ, r)-cumulants simply consists of coefficients of either the polynomial f (an $\alpha_{(i_1,\ldots,i_n)}$ from Equation (1.3)) or the polynomial g (a $\beta_{(i_1,\ldots,i_n)}$ from Equation (1.5)). For the statement of our theorem it is convenient to use a unified notation for A_i's and B_i's:

$$A_i =: C_{i;\ell} \text{ and } B_i =: C_{i;r}, \quad \text{for } 1 \leq i \leq d.$$

Theorem. *Consider all the notations pertaining to $A_1, \ldots, A_d, B_1, \ldots, B_d$ that were introduced above. Let n be a positive integer and let $\chi = (h_1, \ldots, h_n)$ be in $\{\ell, r\}^n$. Let us record explicitly where are the occurrences of ℓ and of r in χ:*

$$\begin{cases} \{m \mid 1 \leq m \leq n,\, h_m = \ell\} =: \{m_\ell(1), \ldots, m_\ell(u)\} & \textit{with } m_\ell(1) < \cdots < m_\ell(u), \\ \{m \mid 1 \leq m \leq n,\, h_m = r\} =: \{m_r(1), \ldots, m_r(v)\} & \textit{with } m_r(1) < \cdots < m_r(v). \end{cases}$$

Then for every $i_1, \ldots, i_n \in \{1, \ldots, d\}$ we have

$$\kappa_\chi(C_{i_1;h_1}, \ldots, C_{i_n;h_n}) = \begin{cases} \alpha_{(i_{m_r(v)}, \ldots, i_{m_r(1)}, i_{m_\ell(1)}, \ldots, i_{m_\ell(u)})}, & \textit{if } h_n = \ell, \\ \beta_{(i_{m_\ell(u)}, \ldots, i_{m_\ell(1)}, i_{m_r(1)}, \ldots, i_{m_r(v)})}, & \textit{if } h_n = r. \end{cases} \tag{1.8}$$

Remark. The above theorem generalizes the result from [3] which says that the canonical left d-tuple $(A_1, \ldots, A_d)$ has R-transform $R_{(A_1,\ldots,A_d)} = f$. Indeed, if the tuple χ from the theorem is set to be $(\ell, \ldots, \ell) \in \{\ell, r\}^n$, then Equation (1.8) says that $\kappa_n(A_{i_1}, \ldots, A_{i_n}) = \alpha_{(i_1,\ldots,i_n)}, \quad \forall\, 1 \leq i_1, \ldots, i_n \leq d$. So then

$$\begin{aligned} R_{(A_1,\ldots,A_d)}(z_1, \ldots, z_d) &:= \sum_{n=1}^{\infty} \sum_{i_1,\ldots,i_n=1}^{d} \kappa_n(A_{i_1}, \ldots, A_{i_n}) z_{i_1} \cdots z_{i_n} \\ &= \sum_{n=1}^{\infty} \sum_{i_1,\ldots,i_n=1}^{d} \alpha_{(i_1,\ldots,i_n)} z_{i_1} \cdots z_{i_n} \\ &= f(z_1, \ldots, z_d), \end{aligned}$$

as claimed. (By setting $\chi = (r, \ldots, r)$ we could, of course, also infer from the above theorem that $R_{(B_1,\ldots,B_d)} = g$.)

Example. The upshot of the theorem is that every mixed moment of the A_i's and B_i's is written as a straight sum (no signs or coefficients!) indexed by $\mathcal{P}^{(\chi)}(n)$, where every term of the sum is a product of α's and β's. For a concrete example, say that we are interested in the mixed moment $\varphi_{\text{vac}}(A_{i_1}B_{i_2}A_{i_3}B_{i_4})$, for some given $i_1, i_2, i_3, i_4 \in \{1, \dots, d\}$. In the above theorem we take $n = 4$ and $\chi = (\ell, r, \ell, r)$, and get:

$$\begin{aligned}\varphi_{\text{vac}}(A_{i_1}B_{i_2}A_{i_3}B_{i_4}) &= \kappa_{(\ell,r,\ell,r)}(A_{i_1}, B_{i_2}, A_{i_3}, B_{i_4}) + \kappa_{(\ell)}(A_{i_1})\, \kappa_{(r,\ell,r)}(B_{i_2}, A_{i_3}, B_{i_4}) \\ &\quad + \cdots + \kappa_\ell(A_{i_1})\, \kappa_r(B_{i_2})\, \kappa_\ell(A_{i_3})\, \kappa_r(B_{i_4}) \\ &= \beta_{(i_3,i_1,i_2,i_4)} + \alpha_{(i_1)}\, \beta_{(i_3,i_2,i_4)} + \cdots + \alpha_{(i_1)}\, \beta_{(i_2)}\, \alpha_{(i_3)}\, \beta_{(i_4)},\end{aligned}$$

a sum of 14 terms corresponding to the 14 partitions in $\mathcal{P}^{(\ell,r,\ell,r)}(4)$. (As is easily checked, $\mathcal{P}^{(\ell,r,\ell,r)}(4)$ is different from $NC(4)$ – it contains the crossing partition $\{\{1,3\}, \{2,4\}\}$, and misses the non-crossing partition $\{\{1,4\}, \{2,3\}\}$.)

Note that the nice feature of getting a summation formula without signs and coefficients is due precisely to the fact that we are using the tuple $\chi = (\ell, r, \ell, r)$ coming from the mixed moment that we want to calculate. If we tried for instance to evaluate the same mixed moment via a sum indexed by $\mathcal{P}^{(r,r,r,r)}(4) = NC(4)$ (which would just be the usual free cumulant expansion), then we would run from the very beginning into the term

$$\begin{aligned}\kappa_{(r,r,r,r)}(A_{i_1}, B_{i_2}, A_{i_3}, B_{i_4}) &= \kappa_4(A_{i_1}, B_{i_2}, A_{i_3}, B_{i_4}) \\ &= \beta_{(i_3,i_1,i_2,i_4)} + \alpha_{(i_1,i_3)}\beta_{(i_2,i_4)} - \alpha_{(i_2,i_3)}\beta_{(i_1,i_4)},\end{aligned}$$

and the nice structure in the formula for $\varphi_{\text{vac}}(A_{i_1}B_{i_2}A_{i_3}B_{i_4})$ (a straight sum of products) would only emerge after going through some more complicated expressions, and doing cancellations between terms.

1.5 Vanishing mixed (ℓ, r)-cumulants, and a question.

Suppose the polynomials f and g considered in section 1.2 are of the form

$$\begin{cases} f(z_1, \dots, z_d) = f_1(z_1) + \cdots + f_d(z_d) \quad \text{and} \\ g(z_1, \dots, z_d) = g_1(z_1) + \cdots + g_d(z_d), \end{cases} \tag{1.9}$$

where $f_1, \dots, f_d, g_1, \dots, g_d$ are polynomials of one variable. Then the formulas defining the canonical operators $A_1, \dots, A_d, B_1, \dots, B_d$ simplify to

$$\begin{cases} A_i = L_i^* \big(I + f_i(L_i)\big), \\ B_i = R_i^* \big(I + g_i(R_i)\big), \end{cases} \quad 1 \le i \le d. \tag{1.10}$$

The d pairs of operators $(A_1, B_1), \dots, (A_d, B_d)$ appearing in Equations (1.10) give an example of bi-free family of pairs of elements of a noncommutative probability space, in the sense of Voiculescu [8]. On the other hand, in view of the formula for (ℓ, r)-cumulants provided by Equation (1.8), the special case of f, g considered in (1.9) can be equivalently described via the requirement that

$$\begin{cases} \kappa_\chi\big(C_{i_1;h_1}, \dots, C_{i_n;h_n}\big) = 0 \\ \quad \text{whenever } \exists\, 1 \le p < q \le n \text{ such that } i_p \ne i_q. \end{cases}$$

This coincidence is in line with the fact that various forms of independence for noncommutative random variables which are considered in the literature have a combinatorial incarnation expressed in terms of the vanishing of some mixed cumulants. It is in fact tempting to make a definition and ask a question, as follows.

Definition. Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space, and let $\big(\kappa_\chi : \mathcal{A}^n \to \mathbb{C} \big)_{n \geq 1, \chi \in \{\ell, r\}^n}$ be the family of (ℓ, r)-cumulant functionals of $(\mathcal{A}, \varphi)$. Let $a_1, b_1, \ldots, a_d, b_d$ be in $\mathcal{A}$. We say that the pairs $(a_1, b_1), \ldots, (a_d, b_d)$ are *combinatorially-bi-free* to mean that the following condition is fulfilled: denoting $c_{i;\ell} := a_i$ and $c_{i;r} := b_i$ for $1 \leq i \leq d$, one has

$$\begin{cases} \kappa_\chi \big(c_{i_1;h_1}, \ldots, c_{i_n;h_n} \big) = 0 \\ \quad \text{whenever } n \geq 2,\ i_1, \ldots, i_n \in \{1, \ldots, d\},\ \chi = (h_1, \ldots, h_n) \in \{\ell, r\}^n \\ \quad \text{and } \exists\, 1 \leq p < q \leq n \text{ such that } i_p \neq i_q. \end{cases} \tag{1.11}$$

Question. Is it true that combinatorial-bi-freeness is equivalent to the (representation theoretic) concept of bi-freeness introduced by Voiculescu in [8]?

After the first version of the present paper was circulated, the above question was found to have an affirmative answer, in the paper [1] by I. Charlesworth, B. Nelson and P. Skoufranis.

1.6 Organization of the paper.

Besides the present introduction, the paper has five other sections. After some review of background in Section 2, we introduce the sets of partitions $\mathcal{P}^{(\chi)}(n)$ in Section 3, via the idea of examining double-ended queues. The alternative description of $\mathcal{P}^{(\chi)}(n)$ via direct bijection with $NC(n)$ is presented in Section 4. In Section 5 we introduce the family of (ℓ, r)-cumulant functionals κ_χ that are associated to a noncommutative probability space. Finally, in Section 6 we prove the main result of the paper, giving the formula for (ℓ, r)-cumulants of canonical operators that was announced in Equation (1.8) above.

2. Background on partitions and on Lukasiewicz paths

Definition 2.1. *[Partitions of* $\{1, \ldots, n\}$*.]*
Let n be a positive integer.

1^o We will let $\mathcal{P}(n)$ denote the set of all partitions of $\{1, \ldots, n\}$. A partition $\pi \in \mathcal{P}(n)$ is thus a set $\pi = \{V_1, \ldots, V_k\}$ where $V_1, \ldots, V_k$ (called the *blocks* of π) are non-empty sets with $V_i \cap V_j = \emptyset$ for $i \neq j$ and with $\cup_{i=1}^k V_i = \{1, \ldots, n\}$.

2^o On $\mathcal{P}(n)$ we consider the partial order given by reverse refinement; that is, for $\pi, \rho \in \mathcal{P}(n)$ we will write "$\pi \leq \rho$" to mean that for every block $V \in \pi$ there exists a block $W \in \rho$ such that $V \subseteq W$. The minimal and maximal partition with respect to this partial order will be denoted as 0_n and 1_n, respectively:

$$0_n := \Big\{ \{1\}, \ldots, \{n\} \Big\}, \quad 1_n := \Big\{ \{1, \ldots, n\} \Big\}. \tag{2.1}$$

3^o Let τ be a permutation of $\{1, \ldots, n\}$ and let $\pi = \{V_1, \ldots, V_k\}$ be in $\mathcal{P}(n)$. We will use the notation "$\tau \cdot \pi$" for the new partition $\tau \cdot \pi := \{ \tau(V_1), \ldots, \tau(V_k) \} \in \mathcal{P}(n)$.

4^o Let π be a partition in $\mathcal{P}(n)$. By the *opposite* of π we will mean the partition

$$\pi_{\text{opp}} := \tau_o \cdot \pi \in \mathcal{P}(n),$$

where τ_o is the order-reversing permutation of $\{1, \ldots, n\}$ (with $\tau_o(m) = n+1-m$ for every $1 \leq m \leq n$).

5^o A partition $\pi \in \mathcal{P}(n)$ is said to be *non-crossing* when it is not possible to find two distinct blocks $V, W \in \pi$ and numbers $a < b < c < d$ in $\{1, \ldots, n\}$ such that $a, c \in V$ and $b, d \in W$. The set $NC(n)$ of all non-crossing partitions in $\mathcal{P}(n)$ will play a significant role in this paper; for a review of basic some facts about it we refer to Lectures 9 and 10 of [4]. Let us record here that $NC(n)$ is one of the many combinatorial structures counted by Catalan numbers, one has $|\, NC(n) \,| = C_n := (2n)!/n!(n+1)!$ (the n-th Catalan number).

Definition 2.2. *[Lukasiewicz paths.]*

1^o We will consider paths in $\mathbb{Z}^2$ which start at $(0,0)$ and make steps of the form $(1, i)$ with $i \in \mathbb{N} \cup \{-1, 0\}$. Such a path,

$$\lambda = \big((0,0), (1, j_1), (2, j_2), \ldots, (n, j_n) \big), \tag{2.2}$$

is said to be a *Lukasiewicz path* when it satisfies the conditions that $j_m \geq 0$ for every $1 \leq m \leq n$ and that $j_n = 0$.

2^o For every $n \geq 1$, we will use the notation $\mathrm{Luk}(n)$ for the set of all Lukasiewicz paths with n steps. For a path $\lambda \in \mathrm{Luk}(n)$ written as in Equation (2.2), we will refer to the vector

$$\vec{\lambda} = (j_1 - 0, j_2 - j_1, \ldots, j_n - j_{n-1}) \in (\mathbb{N} \cup \{-1, 0\})^n \tag{2.3}$$

as to the *rise-vector* of λ. It is immediate how λ can be retrieved from its rise-vector; moreover, it is immediate that a vector $(q_1, \ldots, q_n) \in (\mathbb{N} \cup \{-1, 0\})^n$ appears as rise-vector $\vec{\lambda}$ for some $\lambda \in \mathrm{Luk}(n)$ if and only if its satisfies the conditions that

$$q_1 + \cdots + q_m \geq 0 \text{ for every } 1 \leq m \leq n, \text{ and } q_1 + \cdots + q_n = 0. \tag{2.4}$$

Remark and Notation 2.3. *[The surjection Ψ and the bijection Φ.]*
Let n be a positive integer.

1^o Let $\pi = \{V_1, \ldots, V_k\}$ be a partition in $\mathcal{P}(n)$. Consider the vector $(q_1, \ldots, q_n) \in (\, \mathbb{N} \cup \{-1, 0\} \,)^n$ where for $1 \leq m \leq n$ we put

$$q_m := \begin{cases} |V_i| - 1, & \text{if } m = \min(V_i) \text{ for an } i \in \{1, \ldots, k\}, \\ -1, & \text{otherwise.} \end{cases} \tag{2.5}$$

It is immediately seen that $(q_1, \ldots, q_n)$ satisfies the conditions listed in (2.4), hence it is the rise-vector of a uniquely determined path $\lambda \in \mathrm{Luk}(n)$.

2^o We will denote by $\Psi : \mathcal{P}(n) \to \mathrm{Luk}(n)$ (also denoted as Ψ_n, if needed to clarify what is n) the map which acts by

$$\Psi(\pi) := \lambda, \quad \pi \in \mathcal{P}(n),$$

with λ obtained out of π via the rise-vector described in (2.5).

3^o The map $\Psi : \mathcal{P}(n) \to \mathrm{Luk}(n)$ introduced above has the remarkable property that its restriction to $NC(n)$ gives a bijection from $NC(n)$ onto $\mathrm{Luk}(n)$; for the verification of this fact, see e.g. [4], Proposition 9.8. We will denote by $\Phi : \mathrm{Luk}(n) \to NC(n)$ the inverse of this bijection. That is: for every $\lambda \in \mathrm{Luk}(n)$, we define $\Phi(\lambda)$ to be the unique partition in $NC(n)$ which has the property that

$$\Psi(\, \Phi(\lambda)\,) = \lambda.$$

The bijection Φ confirms the well-known fact that the set of paths $\mathrm{Luk}(n)$ has the same cardinality C_n (Catalan number) as $NC(n)$.

3. Double-ended queues and the sets of partitions $\mathcal{P}^{(\chi)}(n)$

Definition and Remark 3.1. *[Description of a deque device.]*
We will work with a device called *double-ended queue*, or *deque* for short, which is used in the study of information structures in theoretical computer science (see e.g. [2], Section 2.2). We will think about this device in the way depicted in Figures 1, 2 below, and described as follows. Let n be a fixed positive integer, and suppose we have n labelled balls ① , . . . , ⓝ which have to move from an *input pipe* into an *output pipe* (both depicted vertically in the figures), by going through a *deque pipe* (depicted horizontally). The deque device operates in discrete time: it goes through a sequence of states, recorded at times $t = 0, t = 1, \ldots, t = n$, where at time $t = 0$ all the n balls are in the input pipe and at $t = n$ they are all in the output pipe. Compared to the discussion in Knuth's treatise [2], we will limit the kinds of moves [6] that a deque can do, and we will require that:

$$\left\{ \begin{array}{c} \text{For every } 0 \leq i \leq n-1, \text{ the deque device moves from its} \\ \text{state at time } t = i \text{ to its state at time } t = i+1 \text{ by performing} \\ \text{either a \textit{“left-p move”} or a \textit{“right-p move”}, where } p \in \mathbb{N} \cup \{0\}. \end{array} \right.$$

The description of a left-p move is like this:

$$\left\{ \begin{array}{l} \text{“Take the top } p \text{ balls that are in the input pipe, and insert them} \\ \text{into the deque pipe, from the \textit{left}. Then take the \textit{leftmost} ball in} \\ \text{the deque pipe and insert it at the bottom of the output pipe.”} \end{array} \right.$$

The description of a right-p move is analogous to the one of a left-p move, only that the words “left” and “leftmost” get to be replaced by “right” and respectively “rightmost”.

[6] In [2] the deque may perform any sequence of “insertions and deletions at either end of the queue”, where the term “insertion” designates the operation of moving some balls from the input pipe into the deque pipe, and “deletion” refers to moving some balls from the deque pipe into the output pipe. In the present paper we will only allow the special moves described in Definition 3.1, which match, in some sense, the creation and annihilation performed by canonical operators on a full Fock space.

As is clear from the description of a left-p (or right-p) move, the number $p \in \mathbb{N} \cup \{0\}$ used in the move $(t = i) \rightsquigarrow (t = i+1)$ is subjected to the restriction that there exist p balls (or more) in the input pipe at time $t = i$. Note also the additional restriction that $p = 0$ can be used in the move $(t = i) \rightsquigarrow (t = i + 1)$ only if the device has at least 1 ball in the deque pipe at time $t = i$.

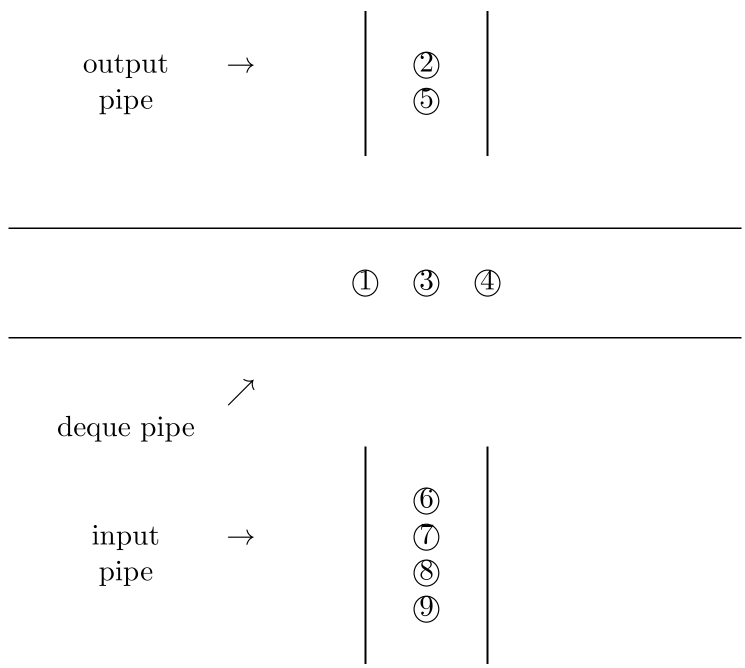

Figure 1: *Say that $n = 9$. Here is a possible state of the deque device at time $t = 2$.*

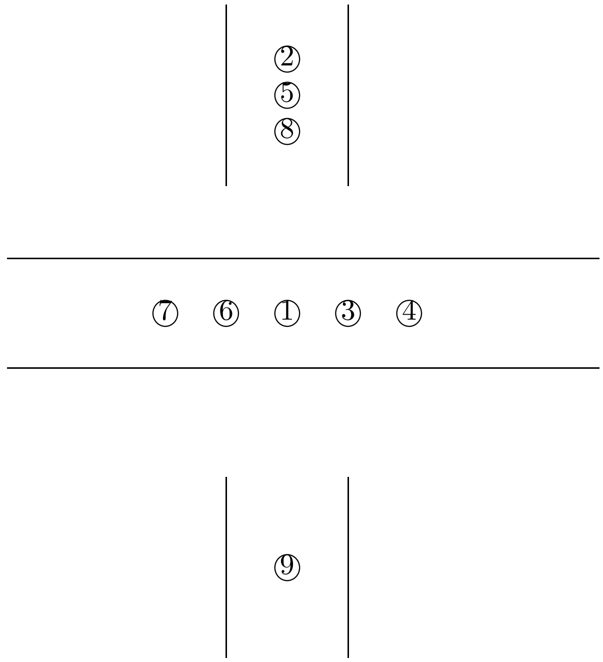

Figure 2: *The deque device from Figure 1, at time* $t = 3$*, after performing a left-*3 *move.*

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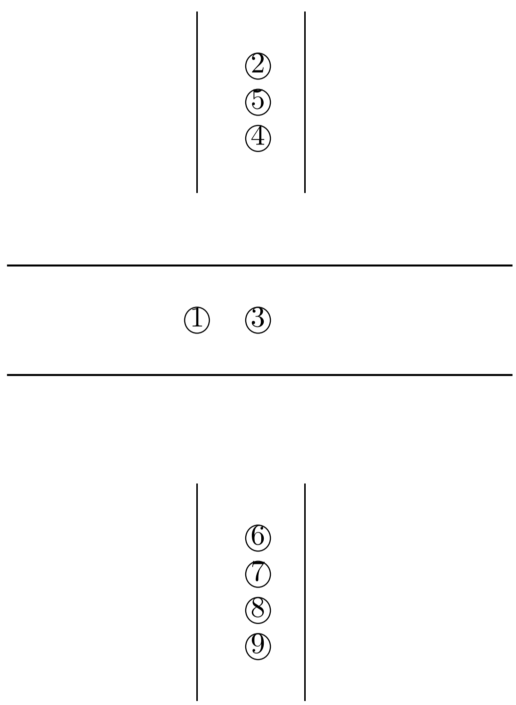

Figure 3: *The deque device from Figure 1, at time* $t = 3$*, after performing a right-*0 *move.*

Definition and Remark 3.2. *[Deque-scenarios.]*
Let n be the same fixed positive integer as in Definition 3.1, and consider the deque device described there. We assume that at time $t = 0$ the n balls are sitting in the input pipe in the order ①, ..., ⓝ, counting top-down. From the description of the moves of the device, it is clear that for every $0 \leq i \leq n$ there are exactly i balls in the output pipe at time $t = i$. In particular, all n balls find themselves in the output pipe at time $t = n$ (even though they may not be sitting in the same order as at time $t = 0$).

We will use the name *deque-scenario* to refer to a possible way of moving the n balls through the deque device, according to the rules described above. Every deque-scenario is thus determined by an array of the form

$$\begin{pmatrix} p_1 & \cdots & p_n \\ h_1 & \cdots & h_n \end{pmatrix}, \tag{3.1}$$

with $p_1, \ldots, p_n \in \mathbb{N} \cup \{0\}$ and $h_1, \ldots, h_n \in \{\ell, r\}$; this array simply records the fact that in order to go from its state at time $t = i-1$ to its state at time $t = i$, the device has executed

$$\begin{cases} \text{a left-}p_i \text{ move,} & \text{if } h_i = \ell \\ \\ \text{a right-}p_i \text{ move,} & \text{if } h_i = r \ , \end{cases} \qquad 1 \leq i \leq n.$$

Let us observe that the top line of the array in (3.1) must satisfy the inequalities

$$\begin{cases} p_1 + \cdots + p_i \geq i, \quad \forall \, 1 \leq i \leq n, \\ \\ \text{where for } i = n \text{ we must have } p_1 + \cdots + p_n = n. \end{cases} \tag{3.2}$$

This is easily seen by counting that at time $t = i$ there are i balls in the output pipe and $n - (p_1 + \cdots + p_i)$ balls in the input pipe, which leaves a difference of

$$n - \Big(i + n - (p_1 + \cdots + p_i) \Big) = (p_1 + \cdots + p_i) - i$$

balls that must be in deque pipe. (And of course, the number of balls found in the deque pipe at $t = i$ must be ≥ 0, with equality for $t = n$.)

It is easy to see that conversely, every array as in (3.1) with $p_1, \ldots, p_n$ satisfying (3.2) will define a working deque-scenario – the inequalities $p_1 + \cdots + p_i \geq i$ ensure that we never run into the situation of having to "move a ball out of the empty deque pipe".

Thus, as a mathematical object, the set of deque-scenarios can be simply introduced as the set of arrays of the kind shown in (3.1), and where (3.2) is satisfied.

Moreover, let us observe that condition (3.2) can be read as saying that the n-tuple

$$(p_1 - 1, \ldots, p_n - 1) \in \big(\mathbb{N} \cup \{-1, 0\} \big)^n \tag{3.3}$$

is the rise-vector of a uniquely determined Lukasiewicz path λ, as reviewed in Section 2. We will then refer to the deque-scenario described by the array (3.1) as the *deque-scenario determined by* (λ, χ), where $\lambda \in \mathrm{Luk}(n)$ has rise-vector given by (3.3) and χ is the n-tuple $(h_1, \ldots, h_n) \in \{\ell, r\}^n$ from the second line of (3.1).

Definition and Remark 3.3. *[Output-time partition associated to a deque-scenario.]* We consider the same notations as above and we look at the deque-scenario determined by (λ, χ), where $\lambda \in \mathrm{Luk}(n)$ has rise-vector $\vec{\lambda} = (p_1 - 1, \ldots, p_n - 1)$ and where $\chi = (h_1, \ldots, h_n) \in \{\ell, r\}^n$.

Let $i \in \{1, \ldots, n\}$ be such that $p_i > 0$, and consider the i-th move of the deque device (the move that takes the device from its state at $t = i-1$ to its state at $t = i$). In that move there is a group of p_i balls (namely those with labels from $p_1 + \ldots + p_{i-1} + 1$ to $p_1 + \ldots + p_i$) which leave together the input pipe. These balls arrive in the output pipe one by one, at various later times, which we record as

$$t_1^{(i)} < t_2^{(i)} < \cdots < t_{p_i}^{(i)}. \tag{3.4}$$

Observe that in particular we have $t_1^{(i)} = i$; indeed, it is also part of the i-th move of the device that the ball with label $p_1 + \cdots + p_i$ goes from the deque pipe into the output pipe. Let us make the notation

$$T_i := \{t_1^{(i)}, \ldots, t_{p_i}^{(i)}\},$$

where $t_1^{(i)}, \ldots, t_{p_i}^{(i)}$ are from (3.4).

In the preceding paragraph we have thus constructed a set $T_i \subseteq \{1, \ldots, n\}$ for every $1 \leq i \leq n$ such that $p_i > 0$. It is clear from the construction that for every such i we have

$$|T_i| = p_i \text{ and } \min(T_i) = i. \tag{3.5}$$

It is also clear that the sets

$$\{T_i \mid 1 \leq i \leq n \text{ such that } p_i > 0\} \tag{3.6}$$

form together a partition of $\pi \in \mathcal{P}(n)$. We will refer to this π as the *output-time partition* associated to the pair (λ, χ).

Example 3.4. 1^o A concrete example: say that $n = 5$, that $\lambda \in \mathrm{Luk}(5)$ has rise-vector $\vec{\lambda} = (2, -1, 1, -1, -1)$, and that $\chi = (r, \ell, \ell, r, \ell)$. In the deque-scenario associated to this pair (λ, χ), there are two groups of balls that are moved from the input pipe into the deque pipe: first group consists of ① , ② , ③ , which arrive in the deque pipe at time $t = 1$; the second group consists of ④ , ⑤ , which arrive in the deque pipe at time $t = 3$. The final order of the balls in the output pipe (counting downwards) is

③ , ① , ⑤ , ② , ④ ,

and the output-time partition associated to (λ, χ) is $\pi = \{\,\{1, 2, 4\},\ \{3, 5\}\,\} \in \mathcal{P}(5)$.

2^o Let n be a positive integer, and consider the n-tuple $\chi_\ell := (\ell, \ldots, \ell) \in \{\ell, r\}^n$. For any $\lambda \in \mathrm{Luk}(n)$, the deque-scenario determined by λ and χ_ℓ is what one might call a "lifo-stack process" (where lifo is a commonly used abbreviation for last-in-first-out). It is easy to see that the output-time partition associated to the pair (λ, χ_ℓ) is the non-crossing partition $\Phi(\lambda)$, where $\Phi : \mathrm{Luk}(n) \to \mathcal{P}(n)$ is as reviewed in Remark 2.3.3.

A similar statement holds if instead of χ_ℓ we use the n-tuple $\chi_r := (r, \ldots, r)$; that is, the output-time partition associated to (λ, χ_r) is the same $\Phi(\lambda) \in NC(n)$ as above.

Definition 3.5. 1^o Consider a pair (λ, χ) where $\lambda \in \text{Luk}(n)$ and $\chi \in \{\ell, r\}^n$, and let $\pi \in \mathcal{P}(n)$ be the output-time partition associated to (λ, χ) in Definition 3.3. We will denote this partition π as "$\Phi_\chi(\lambda)$".

2^o Let χ be an n-tuple in $\{\ell, r\}^n$. The notation introduced in 1^o above defines a function $\Phi_\chi : \text{Luk}(n) \to \mathcal{P}(n)$. We define

$$\mathcal{P}^{(\chi)}(n) := \{\Phi_\chi(\lambda) \mid \lambda \in \text{Luk}(n)\} \subseteq \mathcal{P}(n). \tag{3.7}$$

Proposition 3.6. *Let n be a positive integer, let χ be an n-tuple in $\{\ell, r\}^n$, and consider the function $\Phi_\chi : \text{Luk}(n) \to \mathcal{P}(n)$ introduced in Definition 3.5.*

1^o *Φ_χ is injective, hence it gives a bijection between* Luk(n) *and $\mathcal{P}^{(\chi)}(n)$.*

2^o *Let $\Psi_\chi : \mathcal{P}^{(\chi)}(n) \to \text{Luk}(n)$ be the function inverse to Φ_χ. Then Ψ_χ is the restriction to $\mathcal{P}^{(\chi)}(n)$ of the canonical surjection $\Psi : \mathcal{P}(n) \to \text{Luk}(n)$ that was reviewed in Remark 2.3.2.*

Proof. Both parts of the proposition will follow if we can prove that $\Psi \circ \Phi_\chi$ is the identity map on Luk(n). Thus given a path $\lambda \in \text{Luk}(n)$ and denoting $\Phi_\chi(\lambda) =: \pi$, we have to show that $\Psi(\pi) = \lambda$. But the latter fact is clear from the observation made in (3.5) of Remark 3.3. ■

Remark 3.7. Let n be a positive integer.

1^o From Proposition 3.6 and the fact that $|\text{Luk}(n)| = C_n$ (n-th Catalan number), it follows that $|\mathcal{P}^{(\chi)}(n)| = C_n$ for every $\chi \in \{\ell, r\}^n$.

2^o Suppose that $\chi = (\ell, \ldots, \ell)$. The discussion from Example 3.4.2 shows that in this case we have $\mathcal{P}^{(\chi)}(n) = NC(n)$. Similarly, we also have $\mathcal{P}^{(\chi)}(n) = NC(n)$ in the case when $\chi = (r, \ldots, r)$.

3^o If $n \leq 3$, then it is clear from cardinality considerations that $\mathcal{P}^{(\chi)}(n) = \mathcal{P}(n) = NC(n)$, no matter what $\chi \in \{\ell, r\}^n$ we consider.

For $n \geq 4$, cardinality considerations now show that $\mathcal{P}^{(\chi)}(n)$ is a proper subset of $\mathcal{P}(n)$. It is usually different from $NC(n)$. (For instance the output-time partition from Example 3.4.1 is not in $NC(5)$, showing that $\chi = (r, \ell, \ell, r, \ell) \in \{\ell, r\}^5$ has $\mathcal{P}^{(\chi)}(5) \neq NC(5)$.) Some general properties of the sets of partitions $\mathcal{P}^{(\chi)}(n)$ will follow from their alternative description provided in the next section.

4. An alternative description for $\mathcal{P}^{(\chi)}(n)$

In this section we put into evidence a bijection between $\mathcal{P}^{(\chi)}(n)$ and $NC(n)$ which is implemented by the action of a special permutation σ_χ of $\{1, \ldots, n\}$. The main result of the section is Theorem 4.10. We will arrive to it by observing a certain construction of partition in $NC(n)$ – the "combined-standings partition" associated to a pair $(\lambda, \chi) \in \text{Luk}(n) \times \{\ell, r\}^n$, which is introduced in Definition 4.3.

Definition 4.1. Consider a pair (λ, χ) where $\lambda \in \mathrm{Luk}(n)$ and $\chi = (h_1, \ldots, h_n) \in \{\ell, r\}^n$, and let $\pi \in \mathcal{P}(n)$ be the output-time partition associated to (λ, χ) in Definition 3.3. Let us record explicitly where are the occurrences of ℓ and of r in χ:

$$\begin{cases} \{m \mid 1 \leq m \leq n,\, h_m = \ell\} =: \{m_\ell(1), \ldots, m_\ell(u)\} & \text{with } m_\ell(1) < \cdots < m_\ell(u), \\ \{m \mid 1 \leq m \leq n,\, h_m = r\} =: \{m_r(1), \ldots, m_r(v)\} & \text{with } m_r(1) < \cdots < m_r(v). \end{cases} \tag{4.1}$$

1^o Suppose that in (4.1) we have $u \neq 0$. We define a partition $\rho_{\lambda,\chi;\ell} \in \mathcal{P}(u)$ by the following prescription: two numbers $q, q' \in \{1, \ldots, u\}$ are in the same block of $\rho_{\lambda,\chi;\ell}$ if and only if the numbers $m_\ell(q), m_\ell(q') \in \{1, \ldots, n\}$ belong to the same block of π. The partition $\rho_{\lambda,\chi;\ell}$ will be called the *left-standings partition* associated to (λ, χ).

2^o Likewise, if in (4.1) we have $v \neq 0$, then we define a partition $\rho_{\lambda,\chi;r} \in \mathcal{P}(v)$ via the prescription that $q, q' \in \{1, \ldots, v\}$ belong to the same block of $\rho_{\lambda,\chi;r}$ if and only if $m_r(q), m_r(q')$ are in the same block of π. The partition $\rho_{\lambda,\chi;r}$ will be called the *right-standings partition* associated to (λ, χ).

Remark and Notation 4.2. Consider the framework of Definition 4.1. It will help the subsequent discussion if at this point we introduce some more terminology, which will also clarify the names chosen above for the partitions $\rho_{\lambda,\chi;\ell}$ and $\rho_{\lambda,\chi;r}$.

1^o Same as in Section 3, we will think of the numbers in $\{1, \ldots, n\}$ as of moments in time. We will say that $t \in \{1, \ldots, n\}$ is a *left-time* (respectively a *right-time*) for χ to mean that $h_t = \ell$ (respectively that $h_t = r$). If t is a left-time for χ, then the unique $q \in \{1, \ldots, u\}$ such that $t = m_\ell(q)$ will be called the *left-standing of t in χ*. Likewise, if t is a right-time for χ, then the unique $q \in \{1, \ldots, v\}$ such that $t = m_r(q)$ will be called the *right-standing of t in χ*.

2^o Let the rise-vector of λ be $\vec{\lambda} = (p_1 - 1, \ldots, p_n - 1)$, with $p_1, \ldots, p_n \in \mathbb{N} \cup \{0\}$. The numbers in the set

$$I := \{1 \leq i \leq n \mid p_i > 0\} \tag{4.2}$$

will be called *insertion times* for (λ, χ). Recall that the output-time partition π associated to (λ, χ) has its blocks indexed by I; indeed, Equation (3.6) in Definition 3.3 introduces this partition as

$$\pi = \{T_i \mid i \in I\}. \tag{4.3}$$

With a slight abuse of notation, $\rho_{\lambda,\chi;\ell}$ and $\rho_{\lambda,\chi;r}$ from Definition 4.1 can be written as

$$\rho_{\lambda,\chi;\ell} = \{V_i \mid i \in I\} \quad \text{and} \quad \rho_{\lambda,\chi;r} = \{W_i \mid i \in I\} \tag{4.4}$$

where for every $i \in I$ we put

$$V_i := \{1 \leq q \leq u \mid m_\ell(q) \in T_i\} \text{ and } W_i := \{1 \leq q \leq v \mid m_r(q) \in T_i\}. \tag{4.5}$$

(Every V_i is a block of $\rho_{\lambda,\chi;\ell}$ unless $V_i = \emptyset$, and every W_i is a block of $\rho_{\lambda,\chi;r}$ unless $W_i = \emptyset$. Note that V_i and W_i cannot be empty at the same time, since $|V_i| + |W_i| = |T_i| = p_i > 0$.)

Definition 4.3. We continue to consider the framework of Definition 4.1 and of Notation 4.2. For every $i \in I$ let us denote

$$(n+1) - W_i := \{n+1-q \mid q \in W_i\} \subseteq \{u+1, \ldots, n\}. \tag{4.6}$$

The partition

$$\rho_{\lambda,\chi} := \{V_i \cup (\,(n+1) - W_i\,) \mid i \in I\} \tag{4.7}$$

will be called the *combined-standings partition* associated to (λ, χ).

Remark 4.4. The blocks of the partition $\rho_{\lambda,\chi}$ are indexed by the same set I of insertion times that was used to index the blocks of the output-times partition $\pi = \{T_i \mid i \in I\}$ in Notation 4.2. Moreover, we have

$$|V_i \cup (\,(n+1) - W_i\,)| = |V_i| + |W_i| = |T_i|, \quad \forall\, i \in I;$$

this shows that it must be possible to go between π and $\rho_{\lambda,\chi}$ via the action of some suitably chosen permutation of $\{1, \ldots, n\}$. We next make the easy yet significant observation that the permutation in question can be picked so that it only depends on χ (even though each of π and $\rho_{\lambda,\chi}$ depends not only on χ, but also on λ).

Definition 4.5. Let χ be a tuple in $\{\ell, r\}^n$. We associate to χ a permutation σ_χ of $\{1, \ldots, n\}$ defined (in two-line notation for permutations) as

$$\sigma_\chi := \begin{pmatrix} 1 & \cdots & u & u+1 & \cdots & n \\ m_\ell(1) & \cdots & m_\ell(u) & m_r(v) & \cdots & m_r(1) \end{pmatrix}, \tag{4.8}$$

where $m_\ell(1) < \cdots < m_\ell(u)$ and $m_r(1) < \cdots < m_r(v)$ are as in Definition 4.1 (the lists of occurrences of "ℓ" and "r" in χ).

In (4.8) we include the possibility that $v = 0$ (when $u = n$ and σ_χ is the identity permutation), or that $u = 0$ (when $v = n$ and $\sigma_\chi(m) = n+1-m$ for every $1 \leq m \leq n$).

Lemma 4.6. *Consider a pair $(\lambda, \chi) \in \mathrm{Luk}(n) \times \{\ell, r\}^n$, and let $\pi \in \mathcal{P}(n)$ be the output-time partition associated to (λ, χ) in Definition 3.3. We have*

$$\sigma_\chi \cdot \rho_{\lambda,\chi} = \pi, \tag{4.9}$$

where $\rho_{\lambda,\chi}$ and σ_χ are as in Definitions 4.3 and 4.5, respectively, and where the action of a permutation on a partition is as reviewed in Definition 2.1.3.

Proof. We use the notations established earlier in this section. Clearly, (4.9) will follow if we prove that

$$\sigma_\chi\big(\,V_i \cup (\,(n+1) - W_i\,)\,\big) = T_i, \quad \forall\, i \in I. \tag{4.10}$$

Let us fix an $i \in I$ for which we verify that (4.10) holds. Since the sets $V_i \cup ((n+1) - W_i))$ and T_i have the same cardinality, it suffices to verify the inclusion "$\subseteq$" of the equality. And indeed, referring to how the permutation σ_χ is defined in Equation (4.8), we have:

$$q \in V_i \ \Rightarrow \ \sigma_\chi(q) = m_\ell(q) \in T_i, \quad \text{and}$$

$$q \in (n+1) - W_i \ \Rightarrow \ \sigma_\chi(q) = m_r(n+1-q) \in T_i$$

(where the fact that $m_r(n+1-q) \in T_i$ comes from Equation (4.5), used for the element $n+1-q \in W_i$). Thus both $\sigma_\chi(V_i)$ and $\sigma_\chi((n+1) - W_i)$ are subsets of T_i, and (4.10) follows. ■

Example 4.7. Consider (same as in Example 3.4.1) the concrete case when $n = 5$, $\chi = (r, \ell, \ell, r, \ell)$, and $\lambda \in \mathrm{Luk}(5)$ has rise-vector $\vec{\lambda} = (2, -1, 1, -1, -1)$. As found in Example 3.4.1, the output-time partition associated to this (λ, χ) is $\pi = \{ \{1, 2, 4\}, \{3, 5\} \}$. The set of insertion times for (λ, χ) of this example is $I = \{1, 3\}$; in order to illustrate the system of notation from Equation (4.3), we then write π as

$$\pi = \{T_1, T_3\}, \quad \text{with } T_1 = \{1, 2, 4\} \text{ and } T_3 = \{3, 5\}.$$

The left-times for χ are $m_\ell(1) = 2, m_\ell(2) = 3, m_\ell(3) = 5$, and the right-times are $m_r(1) = 1, m_r(2) = 4$. Since $m_\ell(1) \in T_1$ and $m_\ell(2), m_\ell(3) \in T_3$, we get (in reference to the notations from Equations (4.4) and (4.5)) that

$$V_1 = \{1\}, V_3 = \{2, 3\}, \text{ hence } \rho_{\lambda,\chi;\ell} = \{ \{1\}, \{2, 3\} \} \in \mathcal{P}(3).$$

For the right-times we have $m_r(1), m_r(2) \in T_1$, giving us that

$$W_1 = \{1, 2\}, W_3 = \emptyset, \text{ hence } \rho_{\lambda,\chi;r} = \{ \{1, 2\} \} \in \mathcal{P}(2).$$

The combined-standings partition $\rho_{\lambda,\chi}$ associated to (λ, χ) has blocks

$$V_1 \cup (6 - W_1) = \{1\} \cup \{4, 5\} \ \text{ and } \ V_3 \cup (6 - W_3) = \{2, 3\} \cup \emptyset,$$

hence $\rho_{\lambda,\chi} = \{ \{1, 4, 5\}, \{2, 3\} \}$.

Finally, the permutation associated to χ is

$$\sigma_\chi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 4 & 1 \end{pmatrix}.$$

As explained in the proof of Lemma 4.6, we have that $\sigma_\chi(\{1, 4, 5\}) = \{1, 2, 4\} = T_1$ and $\sigma_\chi(\{2, 3\}) = \{3, 5\} = T_3$, leading to the equality $\sigma_\chi \cdot \rho_{\lambda,\chi} = \pi$.

Our next goal is to prove that the combined-standings partition $\rho_{\lambda,\chi}$ always is a non-crossing partition. In order to obtain this, we first prove a lemma.

Lemma 4.8. *Consider the framework and notations of Definition 4.3. Let us denote the maximal element of I by j, and let us consider the block $S = V_j \cup \big((n+1) - W_j \big)$ of the partition $\rho_{\lambda,\chi}$. Then S is an interval-block (i.e. $S = [t', t''] \cap \mathbb{Z}$ for some $t' \leq t''$ in $\{1, \ldots, n\}$).*

Proof. The conclusion of the lemma is clear if $|S| = 1$, so we will assume that $|S| \geq 2$, i.e. that $p_j \geq 2$.

The maximal insertion time j considered in the lemma is either a left-time or a right-time for χ. We will write the proof by assuming that j is a left-time (the case of a right-time is analogous). We denote the left-standing of the time j as q; recall from Notation 4.2 that this amounts to $j = m_\ell(q)$.

In view of the above assumptions, the deque-scenario associated to (λ, χ) has the following feature: in the j-th move of the deque device, the last p_j balls of the input pipe (with labels between $1 + \sum_{i=1}^{j-1} p_i$ and n) are inserted into deque pipe from the left, and during the same move, the ball n goes into the output pipe. Thus the configuration of balls residing in the deque-pipe at time j is

$$\overline{\underline{\; n' \quad n'' \quad \cdots \quad s \quad x \quad \cdots \quad y \;}} \quad , \tag{4.11}$$

where $n' = n-1, n'' = n-2, \ldots, s = 1 + \sum_{i=1}^{j-1} p_i$, and where "$x, \ldots, y$" is the (possibly empty) configuration of balls that were in the deque pipe at time $j-1$. Let us also note here that each of the remaining moves of the deque device ($(j+1)$-th move up to n-th move) is either a left-0 move or a right-0 move, since the input pipe was emptied at the j-th move.

Due to our assumption that j is a left-time, it is certain that $V_j \neq \emptyset$ (we have in any case that $V_j \ni q$). But W_j may be empty, and we will discuss separately two cases.

Case 1. $W_j = \emptyset$.
In this case all the balls $n', \ldots, s$ exit the deque-pipe by its left side. Some of the balls $x, \ldots, y$ may also exit the deque-pipe by its left side, but they can only do so after all of $n', \ldots, s$ are out of the way. This immediately implies that the times when $n', \ldots, s$ exit the deque-pipe must have consecutive [7] left-standings. It follows that in this case we have $S = V_j = \{q, q+1, \ldots, q + p_j - 1\}$, and hence S is an interval-block of $\rho_{\lambda,\chi}$.

Case 2. $W_j \neq \emptyset$.
In this case some of the balls $n', \ldots, s$ (at least one and at most $p_j - 1$ of them) exit the deque-pipe by its right side. We observe it is not possible to find $s \leq a < b \leq n-1$ such that the ball a exits the deque-pipe by its left side while b exits by the right-side. (Indeed, assume by contradiction that this would be the case. In the picture

$$\overline{\underline{\; n' \quad \cdots \quad b \quad \cdots \quad a \quad \cdots \quad s \quad x \quad \cdots \quad y \;}} \quad ,$$

one of the two balls a, b must be the first to exit the deque-pipe – but that's not possible, since the other ball will block it.) As a consequence, there must exist a label $c \in \{s, \ldots, n-1\}$ such that the balls $s, \ldots, c$ (i.e. the balls with labels in $[s, c] \cap \mathbb{Z}$) exit the deque-pipe by the right side, while the balls with labels in $(c, n-1] \cap \mathbb{Z}$ (if any) exit by the left side.

We next observe that all the balls $x, \ldots, y$ from the picture in (4.11) must exit the deque-pipe by its right side. This follows via the same kind of "blocking" argument as in

[7] Note that the times themselves when $n', \ldots, s$ exit the deque-pipe don't have to be consecutive, because they may be interspersed with some right-times used by balls from $x, \ldots, y$. The "consecutive" claim is only in reference to left-standings.

the preceding paragraph. (Say e.g. that ⓧ wants to exit by the left – then out of the two balls ⓢ and ⓧ, none can be the first to exit the deque-pipe, because it would be blocked by the other.)

Based on the above tallying of how the balls from the picture in (4.11) exit the deque-pipe, a moment's thought shows that the set $W_j \subseteq \{1, \ldots, v\}$ must consist of the $c-s+1$ largest numbers in $\{1, \ldots, v\}$ and that, likewise, the set V_j must be the sub-interval $\{q, \ldots, u\}$ of $\{1, \ldots, u\}$. Then $(n+1) - W_j$ comes to $\{u+1, \ldots, u+(c-s+1)\}$, and the union $S = V_j \cup ((n+1) - W_j)$ is an interval-block of $\rho_{\lambda,\chi}$, as required. ■

Proposition 4.9. *Let n be a positive integer, and let (λ, χ) be a pair in $\mathrm{Luk}(n) \times \{\ell, r\}^n$. The combined-standings partition $\rho_{\lambda,\chi}$ introduced in Definition 4.3 is in $NC(n)$.*

Proof. We proceed by induction on n. The base case $n = 1$ is clear, so we focus on the induction step: we fix an integer $n \geq 2$, we assume the statement of the proposition holds for pairs in $\mathrm{Luk}(m) \times \{\ell, r\}^m$ whenever $1 \leq m \leq n-1$, and we prove that it also holds for pairs in $\mathrm{Luk}(n) \times \{\ell, r\}^n$.

Let us then fix a pair (λ, χ) in $\mathrm{Luk}(n) \times \{\ell, r\}^n$, for which we will prove that $\rho_{\lambda,\chi}$ is in $NC(n)$. We denote $\chi = (h_1, \ldots, h_n)$, and we denote the rise-vector of λ as $\vec{\lambda} = (p_1 - 1, \ldots, p_n - 1)$. Besides $\rho_{\lambda,\chi}$, we will also work with the output-time partition $\pi \in \mathcal{P}(n)$ associated to (λ, χ), and we will use the same notations as earlier in the section:

$$\rho_{\lambda,\chi} = \{V_i \cup (n+1) - W_i \mid i \in I\} \text{ and } \pi = \{T_i \mid i \in I\},$$

where $I = \{1 \leq i \leq n \mid p_i > 0\}$, the set of insertion times for (λ, χ). We will assume that $|I| \geq 2$ (if $|I| = 1$ then clearly $\rho_{\lambda,\chi} = 1_n \in NC(n)$). Same as in Lemma 4.8, we put $j := \max(I)$; we thus have $p_j \geq 1$ and $p_{j+1} = \cdots = p_n = 0$.

Let us put $m := n - p_j = \sum_{i=1}^{j-1} p_i$. Then $m > 0$ (because the assumption $|I| \geq 2$ means there exists $i < j$ with $p_i > 0$), and also $m < n$ (since $p_j > 0$). We consider the m-tuple

$$\chi_o := \chi \mid (\{1, \ldots, n\} \setminus T_j) \in \{\ell, r\}^m$$

(that is, $\chi_o = (h_{t_1}, \ldots, h_{t_m})$, where one writes $\{1, \ldots, n\} \setminus T_j = \{t_1, \ldots, t_m\}$ with $t_1 < \cdots < t_m$). On the other hand, let us consider the Lukasiewicz path $\lambda_o \in \mathrm{Luk}(m)$ determined by the requirement that

$$\vec{\lambda_o} = (p_1 - 1, \ldots, p_n - 1) \mid (\{1, \ldots, n\} \setminus T_j).$$

It is easily seen that the combined-standings partition $\rho_{\lambda_o,\chi_o} \in \mathcal{P}(m)$ associated to (λ_o, χ_o) is obtained from $\rho_{\lambda,\chi} \in \mathcal{P}(n)$ by removing the block $V_j \cup ((n+1) - W_j)$ of $\rho_{\lambda,\chi}$, and then by re-naming the elements of the remaining blocks of $\rho_{\lambda,\chi}$ in increasing order. (Indeed, for this verification all one needs to do is ignore the last group of p_j balls which moves through the pipes of the deque device, in the deque-scenario determined by (λ, χ).)

Now, the block $V_j \cup ((n+1) - W_j)$ removed out of $\rho_{\lambda,\chi}$ is an interval-block, by Lemma 4.8. On the other hand, the partition ρ_{λ_o,χ_o} is in $NC(m)$, due to our induction hypothesis. Thus the partition $\rho_{\lambda,\chi} \in \mathcal{P}(n)$ is obtained via the insertion of an interval-block with $p_j (= n-m)$ elements into a partition from $NC(m)$. This way of looking at $\rho_{\lambda,\chi}$ readily implies that $\rho_{\lambda,\chi} \in NC(n)$, and concludes the proof. ■

It is now easy to prove the main result of this section, which is stated as follows.

Theorem 4.10. *Let χ be a tuple in $\{\ell, r\}^n$, and let the set of partitions $\mathcal{P}^{(\chi)}(n) \subseteq \mathcal{P}(n)$ be as in Definition 3.5. Then $\mathcal{P}^{(\chi)}(n)$ can also be obtained as*

$$\mathcal{P}^{(\chi)}(n) = \big\{ \sigma_{\chi} \cdot \pi \mid \pi \in NC(n) \big\} \subseteq \mathcal{P}(n), \tag{4.12}$$

with σ_χ as in Definition 4.5.

Proof. We will show, equivalently, that $\big\{ \sigma_{\chi}^{-1} \cdot \pi \mid \pi \in \mathcal{P}^{(\chi)}(n) \big\} = NC(n)$. Since on both sides of the latter equality we have sets of the same cardinality C_n, it suffices to verify the inclusion "$\subseteq$". But "$\subseteq$" is clear from Lemma 4.6 and Proposition 4.9, since for $\pi = \Phi_\chi(\lambda)$ with $\lambda \in \mathrm{Luk}(n)$ we get $\sigma_{\chi}^{-1} \cdot \pi = \rho_{\lambda,\chi} \in NC(n)$. ■

Corollary 4.11. *Let n be a positive integer and let χ be a tuple in $\{\ell, r\}^n$.*

1^o *$\mathcal{P}^{(\chi)}(n)$ contains the partitions 0_n and 1_n (from Notation 2.1.2), and also contains all the partitions $\pi \in \mathcal{P}(n)$ which have $n-1$ blocks.*

2^o *The bijection $NC(n) \ni \pi \mapsto \sigma_\chi \cdot \pi \in \mathcal{P}^{(\chi)}(n)$ from Theorem 4.10 is a poset isomorphism, where on both $NC(n)$ and $\mathcal{P}^{(\chi)}(n)$ we consider the partial order "$\leq$" defined by reverse refinement.*

3^o *$(\mathcal{P}^{(\chi)}(n), \leq)$ is a lattice. The meet operation "$\wedge$" of $\mathcal{P}^{(\chi)}(n)$ is described via block-intersections – the blocks of $\pi_1 \wedge \pi_2$ are non-empty intersections $V_1 \cap V_2$, with $V_1 \in \pi_1$ and $V_2 \in \pi_2$.*

Proof. 1^o This follows from the fact that the set $\{0_n, 1_n\} \cup \{\pi \in \mathcal{P}(n) \mid \pi \text{ has } n-1 \text{ blocks}\}$ is contained in $NC(n)$ and is sent into itself by the action of σ_χ (no matter what the permutation σ_χ is).

2^o This is an immediate consequence of the observation that the partial order by reverse refinement is preserved by the action of either σ_χ or σ_χ^{-1}.

3^o The fact that $(\mathcal{P}^{(\chi)}(n), \leq)$ is a lattice follows from 2^o, since $(NC(n), \leq)$ is a lattice. The description of the meet operation of $\mathcal{P}^{(\chi)}(n)$ holds because the meet operation of $NC(n)$ is given by block-intersections, and because the action of σ_χ on partitions respects block-intersections. ■

Remark 4.12. 1^o For every positive integer n, the permutations associated to the (ℓ, r)-words $(\ell, \ldots, \ell)$ and $(r, \ldots, r)$ are

$$\sigma_{(\ell,\ldots,\ell)} := \begin{pmatrix} 1 & 2 & \cdots & n \\ 1 & 2 & \cdots & n \end{pmatrix}, \quad \sigma_{(r,\ldots,r)} := \begin{pmatrix} 1 & 2 & \cdots & n \\ n & n-1 & \cdots & 1 \end{pmatrix}.$$

When plugged into Theorem 4.10, this gives $\mathcal{P}^{(\ell,\ldots,\ell)}(n) = \mathcal{P}^{(r,\ldots,r)}(n) = NC(n)$, a fact that had already been noticed in Remark 3.7.2.

2^o Say that $n = 4$ and that $\chi = (\ell, r, \ell, r)$, with associated partition

$$\sigma_\chi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}.$$

Theorem 4.10 gives, via an easy calculation, that $\mathcal{P}^{(\chi)}(4)$ contains all the partitions of $\{1,2,3,4\}$ with the exception of $\{\ \{1,4\}\, ,\ \{2,3\}\ \}$ (in agreement with the description of this particular $\mathcal{P}^{(\chi)}(n)$ that was mentioned in section 1.2 of the introduction).

In the sequel there will be instances when we will need to "read in reverse" a tuple from $\{\ell, r\}^n$. We conclude the section with an observation about that.

Definition 4.13. For every $n \geq 1$ and $\chi = (h_1, \ldots, h_n) \in \{\ell, r\}^n$, the tuple

$$\chi_{\text{opp}} := (h_n, \ldots, h_1)$$

will be called the *opposite* of χ.

Proposition 4.14. *Let n be a positive integer, let χ be a tuple in $\{\ell, r\}^n$, and consider the opposite tuple χ_{opp}. Then the sets of partitions $\mathcal{P}^{(\chi)}(n)$ and $\mathcal{P}^{(\chi_{\text{opp}})}(n)$ are related by the formula*

$$\mathcal{P}^{(\chi_{\text{opp}})}(n) = \{\pi_{\text{opp}} \mid \pi \in \mathcal{P}^{(\chi)}(n)\}, \tag{4.13}$$

where the opposite π_{opp} of a partition $\pi \in \mathcal{P}(n)$ is as considered in Definition 2.1.4.

Proof. Let τ_o be the order-reversing permutation of $\{1, \ldots, n\}$ that was considered in Definition 2.1.4 ($\tau_o(m) = n + 1 - m$ for $1 \leq m \leq n$). On the other hand let $u \in \{0, 1, \ldots, n\}$ be the number of occurrences of the letter ℓ in the word χ, and let us consider the permutation

$$\tau_u := \begin{pmatrix} 1 & 2 & \cdots & u & u+1 & \cdots & n-1 & n \\ u & u-1 & \cdots & 1 & n & \cdots & u+2 & u+1 \end{pmatrix}. \tag{4.14}$$

(Note that if u happens to be 0, then the permutation τ_0 defined in (4.14) coincides, fortunately, with the permutation τ_o that had been considered above.)

Let π be a partition in $\mathcal{P}(n) \setminus NC(n)$. Let V, W be two distinct blocks of π which cross, and let $a < b < c < d$ be numbers such that $a, c \in V$ and $b, d \in W$. We leave it as an exercise to the reader to check via a case-by-case discussion that the numbers

$$\tau_u(a),\ \tau_u(b),\ \tau_u(c),\ \tau_u(d) \in \{1, \ldots, n\}$$

(despite not being necessarily in increasing order) ensure the existence of a crossing between the blocks $\tau_u(V)$ and $\tau_u(W)$ of the partition $\tau_u \cdot \pi \in \mathcal{P}(n)$.

The argument in the preceding paragraph shows that $\{\tau_u \cdot \pi \mid \pi \in \mathcal{P}(n) \setminus NC(n)\} \subseteq \mathcal{P}(n) \setminus NC(n)$. A cardinality argument forces the latter inclusion to be an equality, and then from the fact that τ_u sends $\mathcal{P}(n)$ bijectively onto itself it also follows that we have

$$\{\tau_u \cdot \pi \mid \pi \in NC(n)\}\ =\ NC(n). \tag{4.15}$$

Now let us consider the positions of the letters ℓ and r in the words χ and χ_{opp}. By tallying these positions and plugging them into the formulas for the permutations σ_χ and $\sigma_{\chi_{\text{opp}}}$ (as in Definition 4.5), one immediately finds that

$$\sigma_{\chi_{\text{opp}}} = \tau_o \, \sigma_\chi \, \tau_u. \tag{4.16}$$

So then we can write:

$$\begin{aligned}
\mathcal{P}^{(\chi_{\text{opp}})}(n) &= \{\sigma_{\chi_{\text{opp}}} \cdot \pi \mid \pi \in NC(n)\} \quad \text{(by Theorem 4.10)} \\
&= \{\tau_o \sigma_\chi \tau_u \cdot \pi \mid \pi \in NC(n)\} \quad \text{(by Eqn.(4.16)} \\
&= \{\tau_o \sigma_\chi \cdot \pi' \mid \pi' \in NC(n)\} \quad \text{(by Eqn.(4.15)} \\
&= \{\tau_o \cdot \pi'' \mid \pi'' \in \mathcal{P}^{(\chi)}(n))\} \quad \text{(by Theorem 4.10)},
\end{aligned}$$

and this establishes the required formula (4.13). ■

5. (ℓ, r)-cumulant functionals

In this section we introduce the family of (ℓ, r)-cumulant functionals associated to a non-commutative probability space. In order to write in a more compressed way the summation formula defining these functionals, we first introduce a notation.

Notation 5.1. *[Restrictions of n-tuples.]*
Let $\mathcal{X}$ be a non-empty set, let n be a positive integer, and let $(x_1, \ldots, x_n)$ be an n-tuple in $\mathcal{X}^n$. For a subset $V = \{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\}$, with $1 \leq m \leq n$ and $1 \leq i_1 < \cdots < i_m \leq n$, we will denote

$$(x_1, \ldots, x_n) \mid V := (x_{i_1}, \ldots, x_{i_m}) \in \mathcal{X}^m.$$

The next definition uses this notation in two ways:

- for $\mathcal{X} = \mathcal{A}$ (algebra of noncommutative random variables);
- for $\mathcal{X} = \{\ell, r\}$, when we talk about the restriction $\chi \mid V$ of a tuple $\chi \in \{\ell, r\}^n$.

Proposition and Definition 5.2. *[(ℓ, r)-cumulants.]*
Let $(\mathcal{A}, \varphi)$ be a nocommutative probability space. There exists a family of multilinear functionals

$$\Big(\kappa_\chi : \mathcal{A}^n \to \mathbb{C} \Big)_{n \geq 1, \, \chi \in \{\ell, r\}^n}$$

which is uniquely determined by the requirement that

$$\left\{ \begin{array}{l} \varphi(a_1 \cdots a_n) = \sum_{\pi \in \mathcal{P}^{(\chi)}(n)} \Big(\prod_{V \in \pi} \kappa_{\chi|V}((a_1, \ldots, a_n) \mid V) \Big), \\ \\ \textit{for every } n \geq 1, \ \chi \in \{\ell, r\}^n \textit{ and } a_1, \ldots, a_n \in \mathcal{A}. \end{array} \right. \tag{5.1}$$

These κ_χ's will be called the (ℓ, r)-cumulant functionals *of $(\mathcal{A}, \varphi)$.*

Proof. For $n = 1$ we define $\kappa_{(\ell)} = \kappa_{(r)} = \varphi$. We then proceed recursively, where for every $n \geq 2$, every $\chi \in \{\ell, r\}^n$ and every $a_1, \ldots, a_n \in \mathcal{A}$ we put

$$\kappa_\chi(a_1, \ldots, a_n) = \varphi(a_1 \cdots a_n) - \sum_{\substack{\pi \in \mathcal{P}^{(\chi)}(n) \\ \pi \neq 1_n}} \Big(\prod_{V \in \pi} \kappa_{\chi | V}(\, (a_1, \ldots, a_n) \mid V \,) \Big). \tag{5.2}$$

It is immediate that (5.2) defines indeed a family of multilinear functionals which fulfil (5.1). The uniqueness part of the proposition is also immediate, by following the (obligatory) recursion (5.2). ■

Remark 5.3. Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space, let $(\kappa_n)_{n=1}^\infty$ be the family of free cumulant functionals of $(\mathcal{A}, \varphi)$, and let $\big(\kappa_\chi : \mathcal{A}^n \to \mathbb{C} \big)_{n \geq 1,\, \chi \in \{\ell, r\}^n}$ be the family of (ℓ, r)-cumulant functionals introduced in Definition 5.2.

1^o As noticed in Remark 3.7.2, one has $\mathcal{P}^{(\ell, \ldots, \ell)}(n) = \mathcal{P}^{(r, \ldots, r)}(n) = NC(n)$. By plugging this fact into the recursion (5.2) which characterizes the functionals κ_χ, one immediately obtains the fact (already advertised in the introduction) that

$$\kappa_{(\underbrace{\ell, \ldots, \ell}_{n})} = \kappa_{(\underbrace{r, \ldots, r}_{n})} = \kappa_n, \quad \forall\, n \geq 1.$$

2^o If $n \leq 3$, then we actually have $\kappa_\chi = \kappa_n$ for every $\chi \in \{\ell, r\}^n$. This comes from the fact, observed in Remark 3.7.3, that $\mathcal{P}^{(\chi)}(n) = NC(n)$ when $n \leq 3$, no matter what $\chi \in \{\ell, r\}^n$ we consider.

3^o For $n \geq 4$, the functionals κ_χ with $\chi \in \{\ell, r\}^n$ are generally different from κ_n. Say for instance that $\chi = (\ell, r, \ell, r) \in \{\ell, r\}^4$, then the difference between the lattices $NC(4)$ and $\mathcal{P}^{(\chi)}(4)$ leads to the fact that for $a_1, \ldots, a_4 \in \mathcal{A}$ we have

$$\begin{aligned} \kappa_{(\ell, r, \ell, r)}(a_1, \ldots, a_4) = \;& \kappa_4(a_1, \ldots, a_4) \\ & + \kappa_2(a_1, a_4)\kappa_2(a_2, a_3) - \kappa_2(a_1, a_3)\kappa_2(a_2, a_4). \end{aligned}$$

Remark 5.4. Let us also record here a formula, concerning (ℓ, r)-cumulants, which is related to the reading of (ℓ, r)-words in reverse (i.e. to looking at χ versus χ_{opp}, as in Definition 4.13 and Proposition 4.14). Suppose that $(\mathcal{A}, \varphi)$ is a $*$-probability space. Then, with $\big(\kappa_\chi : \mathcal{A}^n \to \mathbb{C} \big)_{n \geq 1,\, \chi \in \{\ell, r\}^n}$ denoting the family of (ℓ, r)-cumulant functionals of $(\mathcal{A}, \varphi)$, one has

$$\left\{ \begin{array}{l} \kappa_\chi(a_1^*, \ldots, a_n^*) = \overline{\kappa_{\chi_{\text{opp}}}(a_n, \ldots, a_1)}, \\ \\ \quad \text{for every } n \geq 1,\ \chi \in \{\ell, r\}^n \text{ and } a_1, \ldots, a_n \in \mathcal{A}. \end{array} \right. \tag{5.3}$$

The verification of (5.3) is easily done by induction on n, where one relies on the bijections

$$\mathcal{P}^{(\chi)}(n) \ni \pi \mapsto \pi_{\text{opp}} \in \mathcal{P}^{(\chi_{\text{opp}})}(n), \quad \text{for } n \geq 1 \text{ and } \chi \in \{\ell, r\}^n,$$

that were observed in Proposition 4.14. (The proof of the induction step starts, of course, by writing that $\varphi(a_1^* \cdots a_n^*) = \overline{\varphi(a_n \cdots a_1)}$; each of the moments $\varphi(a_1^* \cdots a_n^*)$ and $\varphi(a_n \cdots a_1)$ is then expanded into (ℓ, r)-cumulants, in the way described in Definition 5.2.)

6. (ℓ, r)-cumulants of canonical operators

In this section we prove the theorem announced in section 1.4 of the Introduction. We will adopt the framework and notations of the theorem – so we are dealing with the d-tuples $(A_1, \ldots, A_d)$ and $(B_1, \ldots, B_d)$ of left and respectively right canonical operators on $\mathcal{T}_d$, which were defined in Equations (1.3)–(1.6) of section 1.2 by starting from two non-commutative polynomials $f(z_1, \ldots, z_d)$ and $g(z_1, \ldots, z_d)$. Recall that the coefficients of $z_{i_1} \cdots z_{i_n}$ in the polynomials f and g are denoted as $\alpha_{(i_1,\ldots,i_n)}$ and as $\beta_{(i_1,\ldots,i_n)}$, respectively.

In the formula claimed by the theorem we used the unified notation

$$A_i =: C_{i;\ell} \text{ and } B_i =: C_{i;r}, \quad \text{for } 1 \leq i \leq d. \tag{6.1}$$

In order to give a concise re-statement of that formula, let us also introduce a unified notation for the relevant coefficients α and β, as follows.

Definition 6.1. *[Bi-words and bi-mixtures of coefficients.]*
Let n be a positive integer.

1^o The elements of the set $\{1, \ldots, d\}^n \times \{\ell, r\}^n$ will be called *bi-words* of length n.

2^o Let $(\omega; \chi)$ be a bi-word of length n, where $\omega = (i_1, \ldots, i_n) \in \{1, \ldots, d\}^n$ and $\chi = (h_1, \ldots, h_n) \in \{\ell, r\}^n$. We denote

$$\gamma(\omega; \chi) := \begin{cases} \alpha_{(i_{m_r(v)},\ldots,i_{m_r(1)},i_{m_\ell(1)},\ldots,i_{m_\ell(u)})}, & \text{if } h_n = \ell, \\ \\ \beta_{(i_{m_\ell(u)},\ldots,i_{m_\ell(1)},i_{m_r(1)},\ldots,i_{m_r(v)})}, & \text{if } h_n = r, \end{cases} \tag{6.2}$$

where $m_\ell(1) < \cdots < m_\ell(u)$ and $m_r(1) < \cdots < m_r(v)$ record the lists of occurrences of ℓ and of r in χ (same convention of notation as in Definition 4.1). We will refer to $\gamma(\omega; \chi)$ as the *bi-mixture* of α's and β's corresponding to the bi-word $(\omega; \chi)$.

The result we want to prove can then be stated as follows.

Theorem 6.2. *For every $n \geq 1$ and every $\chi = (h_1, \ldots, h_n) \in \{\ell, r\}^n$, $\omega = (i_1, \ldots, i_n) \in \{1, \ldots, d\}^n$, one has*

$$\kappa_\chi(C_{i_1;h_1}, \ldots, C_{i_n;h_n}) = \gamma(\omega; \chi). \tag{6.3}$$

The remaining part of the section is devoted to the proof of Theorem 6.2. The proof will go by formalizing, in Lemma 6.6 below, the intuitive idea that the action of $A_1, \ldots, A_d$, $B_1, \ldots, B_d$ on the vacuum vector $\xi_{\text{vac}} \in \mathcal{T}_d$ is closely related to the deque-scenarios from Section 3 of the paper.

In order to state Lemma 6.6, we need the concept (related to the one from Definition 6.1.2) of what is a "reverse-bi-mixture" of coefficients α and β.

Definition 6.3. Let n be a positive integer and let $(\omega;\chi)$ be a bi-word of length n, where $\omega = (i_1, \ldots, i_n)$ and $\chi = (h_1, \ldots, h_n)$. We will denote

$$\widetilde{\gamma}(\omega;\chi) := \begin{cases} \alpha_{(i_{m_r(1)},\ldots,i_{m_r(v)},i_{m_\ell(u)},\ldots,i_{m_\ell(1)})}, & \text{if } h_1 = \ell \\ \\ \beta_{(i_{m_\ell(1)},\ldots,i_{m_\ell(u)},i_{m_r(v)},\ldots,i_{m_r(1)})}, & \text{if } h_1 = r, \end{cases} \tag{6.4}$$

where $m_\ell(1) < \cdots < m_\ell(u)$ and $m_r(1) < \cdots < m_r(v)$ record the lists of occurrences of ℓ and of r in χ (same convention of notation as in Definition 4.1 and in Definition 6.1). We will refer to $\widetilde{\gamma}(\omega;\chi)$ as the *reverse-bi-mixture* of α's and β's corresponding to the bi-word $(\omega;\chi)$.

Remark 6.4. It is obvious that the reverse-bi-mixtures which were just introduced are related to the bi-mixtures from Definition 6.1 by the formula

$$\gamma(\omega;\chi) = \widetilde{\gamma}(\omega_{\text{opp}};\chi_{\text{opp}}), \tag{6.5}$$

where for $\chi = (h_1, \ldots, h_n)$ and $\omega = (i_1, \ldots, i_n)$ we put $\chi_{\text{opp}} := (h_n, \ldots, h_1)$ (same as in Definition 4.13) and $\omega_{\text{opp}} := (i_n, \ldots, i_1)$.

Let us also record an immediate extension of Equation (6.5), namely that for every non-empty set $T \subseteq \{1, \ldots, n\}$ we have

$$\gamma((\omega;\chi) \mid T) = \widetilde{\gamma}(\,(\omega_{\text{opp}};\chi_{\text{opp}}) \mid (n+1) - T\,), \tag{6.6}$$

with $(n+1) - T := \{n+1-t \mid t \in T\}$.
[The restrictions of bi-words that have appeared in Equation (6.6) are defined by the same convention as used in Notation 5.1 – e.g. we have

$$(\omega;\chi) \mid T := (\omega \mid T\,;\, \chi \mid T) \in \{1, \ldots, d\}^m \times \{\ell, r\}^m,$$

where m is the number of elements of T.]

In the statement of Lemma 6.6 we will also use the following notation.

Notation 6.5. We denote

$$\begin{cases} X_{0;\ell} = X_{0;r} = I \quad \text{(identity operator)}; \\ \\ X_{p;\ell} = \sum_{i_1,\ldots,i_p=1}^{d} \alpha_{(i_1,\ldots,i_p)} L_{i_p} \cdots L_{i_1}, \quad \text{for } p \geq 1; \\ \\ X_{p;r} = \sum_{i_1,\ldots,i_p=1}^{d} \beta_{(i_1,\ldots,i_p)} R_{i_p} \cdots R_{i_1}, \quad \text{for } p \geq 1. \end{cases} \tag{6.7}$$

The canonical operators A_i, B_i that we are dealing with can then be written as

$$A_i = L_i^* \sum_{p=0}^{\infty} X_{p;\ell}, \quad B_i = R_i^* \sum_{p=0}^{\infty} X_{p;r}, \quad \text{for } 1 \leq i \leq d. \tag{6.8}$$

(The sums in (6.8) are actually finite, since $X_{p;\ell} = X_{p;r} = 0$ for p large enough.)

It will be convenient to use a "unified left-right notation" of the Equations (6.8), as follows. We already have a unified notation for A_i and B_i (the $C_{i;h}$ from Equation (6.1)), and let us also denote

$$L_i =: S_{i,\ell}, \quad R_i =: S_{i,r}, \quad \text{for } 1 \leq i \leq d.$$

Then (6.8) can be put in the form

$$C_{i;h} = S_{i;h}^* \sum_{p=0}^{\infty} X_{p;h}, \text{ for } 1 \leq i \leq d \text{ and } h \in \{\ell, r\}. \tag{6.9}$$

Lemma 6.6. *Let n be a positive integer, and consider the following items:*

- *an n-tuple $\omega = (i_1, \ldots, i_n) \in \{1, \ldots, d\}^n$;*
- *an n-tuple $\chi = (h_1, \ldots, h_n) \in \{\ell, r\}^n$;*
- *a Lukasiewicz path $\lambda \in \mathrm{Luk}(n)$ with rise-vector denoted as $\vec{\lambda} = (p_1 - 1, \ldots, p_n - 1)$, where $p_1, \ldots, p_n \in \mathbb{N} \cup \{0\}$.*

Let $\pi \in \mathcal{P}^{(\chi)}(n)$ be the output-time partition associated to (λ, χ) in Definition 3.3. Then we have

$$X_{p_1;h_1}^* S_{i_1;h_1} \cdots X_{p_n;h_n}^* S_{i_n;h_n} \xi_{\text{vac}} = \overline{c}\, \xi_{\text{vac}}, \quad \textit{where} \quad c = \prod_{T \in \pi} \widetilde{\gamma}(\,(\omega; \chi) \mid T\,). \tag{6.10}$$

In Equation (6.10), the operators $X_{p;h}$ and $S_{i;h}$ are as in Notation 6.5, and the coefficients $\widetilde{\gamma}$ are reverse-bi-mixtures, as in Definition 6.3.

Proof. Let $\{j_1 < j_2 < \ldots < j_t\} = \{i | p_i > 0\}$ let $\pi = \{T_{j_1}, \ldots, T_{j_t}\}$, where for $r = 1, \ldots, t$, we have that T_{j_r} denotes the block of the output-time partition corresponding to time j_r, i.e., the block whose minimal element is j_r. We abbreviate $k = j_t$.

We proceed by induction on t. We first deal with the base case. If $t = 1$ then we must have $k = 1$, $p_1 = p_k = n$, $p_2 = \ldots = p_n = 0$, and $\pi = \{T_1\} = \{\{1, \ldots, n\}\}$. If we denote $\{m_\ell(1) < \ldots m_\ell(u)\} = \{i | h_i = \ell\}$ and $\{m_r(1) < \ldots < m_r(v)\} = \{i | h_i = r\}$ as in Definition 6.3, then we have

$$\begin{aligned}
&X_{p_1;h_1}^* S_{i_1;h_1} \cdots X_{p_n;h_n}^* S_{i_n;h_n} \xi_{\text{vac}} \\
&\quad = \quad X_{n;h_1}^* \left(S_{i_1;h_1} \cdots S_{i_n;h_n} \xi_{\text{vac}} \right) \\
&\quad = \quad X_{n;h_1}^* e_{m_\ell(1)} \otimes \ldots \otimes e_{m_\ell(u)} \otimes e_{m_r(v)} \otimes \ldots \otimes e_{m_r(1)} \\
&\quad = \quad \overline{\widetilde{\gamma}\left((\omega, \chi) | T_1\right)} \xi_{\text{vac}} \\
&\quad = \quad \overline{c} \xi_{\text{vac}}.
\end{aligned}$$

Now assume that $t > 1$ and that the conclusion of the lemma holds for all smaller values of t. Let

$$f \colon \{1, 2, \ldots, n - p_k\} \to \{1, \ldots, n\} \setminus T_k$$

denote the unique increasing bijection. We abbreviate

$$\widehat{\omega} = \left(i_{f(1)}, \ldots, i_{f(n-p_k)}\right), \quad \widehat{\chi} = \left(h_{f(1)}, \ldots, h_{f(n-p_k)}\right),$$

and we also denote $\vec{v} := \big(p_{f(1)}-1, \ldots, p_{f(n-p_k)-1}\big)$. Let $\widehat{\lambda}$ be the Lukasiewicz path associated to $\vec{v}$, and let $\widehat{\pi} \in \mathcal{P}^{(\widehat{\chi})}(n-p_k)$ be the output-time partition associated to $(\widehat{\lambda}, \widehat{\chi})$. We now note, as is implicit in the discussions in Sections 3 and 4, that

$$f(\widehat{\pi}) = \{T_1, \ldots, T_{j_{t-1}}\}.$$

Details of this observation are left to the reader. Observe now that by the induction hypothesis we have

$$X^*_{p_{f(1)},h_{f(1)}} S_{i_{f(1)};h_{f(1)}} \cdots X^*_{p_{f(n-p_k)},h_{f(n-p_k)}} S_{i_{f(n-p_k)};h_{f(n-p_k)}} = \overline{\widehat{c}}\xi_{\text{vac}},$$

where

$$\widehat{c} = \prod_{T\in\widehat{\pi}} \widetilde{\gamma}(\,(\widehat{\omega};\widehat{\chi}) \mid T\,) = \prod_{T\in\pi, T\neq T_k} \widetilde{\gamma}(\,(\omega;\chi) \mid T\,).$$

Let us list elements of the set $\{k,\ldots,n\}$ as $d_k,\ldots,d_n$ by listing left elements first in the increasing order followed by the right elements in the decreasing order, i.e., the order in the list $d_k,\ldots,d_n$ respects the order from the list $m_\ell(1),\ldots,m_\ell(u), m_r(v),\ldots,m_r(1)$. Now note that we have

$$\begin{aligned}
&X^*_{p_k;h_k} S_{i_k;h_k} \cdots X^*_{p_n;h_n} S_{i_n;h_n}\xi_{\text{vac}} \\
&\quad= X^*_{p_k;h_k}\left(S_{i_k;h_k}\cdots S_{i_n;h_n}\xi_{\text{vac}}\right) \\
&\quad= X^*_{p_k;h_k} e_{d_k}\otimes\ldots\otimes e_{d_n} \\
&\quad= \begin{cases}
\overline{\alpha_{d_n,\ldots,d_k}}\xi_{\text{vac}} & \text{, if } p_k = n-k+1 \text{ and } h_k=\ell \\
\overline{\beta_{d_k,\ldots,d_n}}\xi_{\text{vac}} & \text{, if } p_k = n-k+1 \text{ and } h_k=r \\
\overline{\alpha_{d_{k+p_k-1},\ldots,d_k}} e_{d_{k+p_k}}\otimes\ldots\otimes e_{d_n} & \text{, if } p_k < n-k+1 \text{ and } h_k=\ell \\
\overline{\beta_{d_{n+1-p_k},\ldots,d_n}} e_{d_k}\otimes\ldots\otimes e_{d_{n-p_k}} & \text{, if } p_k < n-k+1 \text{ and } h_k=r
\end{cases} \\
&\quad= \overline{\widetilde{\gamma}\left((\omega,\chi)|T_k\right)} S_{i_{f(k)};d_{f(k)}}\cdots S_{i_{f(n-p_k)};d_{f(n-p_k)}}\xi_{\text{vac}}.
\end{aligned}$$

Hence we have

$$\begin{aligned}
&X^*_{p_1;h_1}S_{i_1;h_1}\cdots X^*_{p_{k-1};h_{k-1}}S_{i_{k-1};h_{k-1}}X^*_{p_k;h_k}S_{i_k;h_k}\cdots X^*_{p_n;h_n}S_{i_n;h_n}\xi_{\text{vac}} \\
&\quad= X^*_{p_1;h_1}S_{i_1;h_1}\cdots X^*_{p_{k-1};h_{k-1}}\left(\overline{\widetilde{\gamma}}\left((\omega,\chi)|T_k\right) S_{i_{f(k)};d_{f(k)}}\cdots S_{i_{f(n-p_k)};d_{f(n-p_k)}}\xi_{\text{vac}}\right) \\
&\quad= \overline{\widetilde{\gamma}\left((\omega,\chi)|T_k\right)} X^*_{p_1;h_1}S_{i_1;h_1}\cdots X^*_{p_{k-1};h_{k-1}} S_{i_{f(k)};d_{f(k)}}\cdots S_{i_{f(n-p_k)};d_{f(n-p_k)}}\xi_{\text{vac}} \\
&\quad= \overline{\widetilde{\gamma}\left((\omega,\chi)|T_k\right)} X^*_{p_{f(1)};h_{f(1)}}S_{i_{f(1)};h_{f(1)}}\cdots X^*_{p_{f(n-p_k)};h_{f(n-p_k)}}S_{i_{f(n-p_k)};h_{f(n-p_k)}} \\
&\quad= \overline{\widetilde{\gamma}\left((\omega,\chi)|T_k\right)}\cdot \prod_{T\in\pi,T\neq T_k}\overline{\widetilde{\gamma}(\,(\omega;\chi)\mid T\,)}\xi_{\text{vac}} \\
&\quad= \prod_{T\in\pi}\overline{\widetilde{\gamma}(\,(\omega;\chi)\mid T\,)}\xi_{\text{vac}} = \overline{c}\xi_{\text{vac}}.
\end{aligned}$$

This conclude the induction step. ■

Example 6.7. For clarity, let us follow the preceding lemma in the concrete case (also discussed earlier, in Examples 3.4.1 and 4.7) where $n=5$, $\chi=(r,\ell,\ell,r,\ell)$, and $\lambda\in\text{Luk}(5)$ has rise-vector $\vec{\lambda}=(2,-1,1,-1,-1)$. As found in Example 3.4.1, the output-time partition associated to this (λ, χ) is $\pi = \{ \{1,2,4\}, \{3,5\} \}$. Let us also fix a tuple $\omega = (i_1, \ldots, i_5) \in \{1, \ldots, d\}^5$. We have $\widetilde{\gamma}\big((\omega; \chi) \mid \{1,2,4\} \big) = \widetilde{\gamma}\big((i_1, i_2, i_4); (r, \ell, r) \big) = \beta_{(i_2, i_4, i_1)}$ and $\widetilde{\gamma}\big((\omega; \chi) \mid \{3,5\} \big) = \widetilde{\gamma}\big((i_3, i_5); (\ell, \ell) \big) = \alpha_{(i_5, i_3)}$. The constant c from Equation (6.10) is thus $c = \beta_{(i_2, i_4, i_1)} \, \alpha_{(i_5, i_3)}$, and the formula claimed by the lemma should come to

$$X^*_{3;r} R_{i_1} X^*_{0;\ell} L_{i_2} X^*_{2;\ell} L_{i_3} X^*_{0;r} R_{i_4} X^*_{0;\ell} L_{i_5} \xi_{\mathrm{vac}} = \overline{c}\, \xi_{\mathrm{vac}}$$

for this particular value of c. And indeed, let us record how ξ_{vac} travels when we apply to it the operators listed on the left-hand side of the above equation: we get

$$\begin{array}{lll}
\xi_{\mathrm{vac}} & \mapsto & L_{i_5} \xi_{\mathrm{vac}} = e_{i_5} \\
 & \mapsto & R_{i_4} e_{i_5} = e_{i_5} \otimes e_{i_4} \\
 & \mapsto & X^*_{2;\ell} L_{i_3} (e_{i_5} \otimes e_{i_4}) = X^*_{2;\ell}\, e_{i_3} \otimes e_{i_5} \otimes e_{i_4} = \overline{\alpha_{i_5, i_3}}\, e_{i_4} \\
 & \mapsto & L_{i_2} (\overline{\alpha_{i_5, i_3}}\, e_{i_4}) = \overline{\alpha_{i_5, i_3}}\; e_{i_2} \otimes e_{i_4} \\
 & \mapsto & X^*_{3;r} R_{i_1} (\overline{\alpha_{i_5, i_3}}\, e_{i_2} \otimes e_{i_4}) = \overline{\alpha_{i_5, i_3}}\, X^*_{3;r}\, (e_{i_2} \otimes e_{i_4} \otimes e_{i_1}) = \overline{\alpha_{i_5, i_3}} \cdot \overline{\beta_{i_2, i_4, i_1}}\, \xi_{\mathrm{vac}},
\end{array}$$

as claimed.

Proposition 6.8. *Let n be a positive integer and let $(\omega; \chi)$ be a bi-word of length n, where $\omega = (i_1, \ldots, i_n)$ and $\chi = (h_1, \ldots, h_n)$. We have*

$$\varphi_{\mathrm{vac}}(C_{i_1;h_1} \cdots C_{i_n;h_n}) = \sum_{\pi \in \mathcal{P}^{(\chi)}(n)} \Big(\prod_{T \in \pi} \gamma\big((\omega; \chi) \mid T \,\big) \Big). \tag{6.11}$$

where the bi-mixtures "γ" on the right-hand side of the equation are as introduced in Definition 6.1.

Proof. Write each of $C_{i_1;h_1}, \ldots, C_{i_n;h_n}$ as a sum in the way indicated in Equation (6.9) of Notation 6.5, then expand the ensuing product of sums; we get

$$\varphi_{\mathrm{vac}}(C_{i_1;h_1} \cdots C_{i_n;h_n}) = \sum_{p_1, \ldots, p_n = 0}^{\infty} \mathrm{term}_{(p_1, \ldots, p_n)}, \tag{6.12}$$

where for every $p_1, \ldots, p_n \in \mathbb{N} \cup \{0\}$ we put

$$\begin{aligned}
\mathrm{term}_{(p_1, \ldots, p_n)} \;:=&\;\; \varphi_{\mathrm{vac}}(S^*_{i_1;h_1} X_{p_1;h_1} \cdots S^*_{i_n;h_n} X_{p_n;h_n}) \qquad\qquad (6.13)\\
=&\;\; \langle S^*_{i_1;h_1} X_{p_1;h_1} \cdots S^*_{i_n;h_n} X_{p_n;h_n}\, \xi_{\mathrm{vac}} \,,\, \xi_{\mathrm{vac}} \rangle.
\end{aligned}$$

We will proceed by examining what n-tuples $(p_1, \ldots, p_n) \in (\mathbb{N} \cup \{0\})^n$ may contribute a non-zero term in the sum from (6.12).

So let $p_1, \ldots, p_n$ be in $\mathbb{N} \cup \{0\}$. We make the following observations.
• If there exists $m \in \{1, \ldots, n\}$ with $p_m + \cdots + p_n < (n+1) - m$, then $\mathrm{term}_{(p_1, \ldots, p_n)} = 0$. Indeed, if such an m exists then it is immediately seen that

$$S^*_{i_m;h_m} X_{p_m;h_m} \cdots S^*_{i_n;h_n} X_{p_n;h_n}\, \xi_{\mathrm{vac}} = 0,$$

which makes the inner product from (6.13) vanish.

• If $p_1 + \cdots + p_n > n$, then $\mathrm{term}_{(p_1,\ldots,p_n)} = 0$. Indeed, in this case the vector $S^*_{i_1;h_1} X_{p_1;h_1} \cdots S^*_{i_n;h_n} X_{p_n;h_n} \, \xi_{\mathrm{vac}}$ is seen to belong to the subspace

$$\mathrm{span}\{ e_{j_1} \otimes \cdots \otimes e_{j_q} \mid 1 \leq j_1, \ldots, j_q \leq d \} \subseteq \mathcal{T}_d,$$

where $q = (p_1 + \cdots + p_n) - n > 0$. The latter subspace is orthogonal to ξ_{vac}, and this again makes the inner product from (6.13) vanish.

The observations made in the preceding paragraph show that a necessary condition for $\mathrm{term}_{(p_1,\ldots,p_n)} \neq 0$ is that

$$\begin{cases} p_m + \cdots + p_n \geq (n+1) - m, \quad \forall \, 1 \leq m \leq n, \\ \\ \text{where for } m = 1 \text{ we must have } p_1 + \cdots + p_n = n. \end{cases}$$

This says precisely that the tuple $(p_n - 1, \ldots, p_1 - 1)$ is the rise-vector of a uniquely determined path $\lambda \in \mathrm{Luk}(n)$. Hence the sum on the right-hand side of (6.12) is in fact, in a natural way, indexed by $\mathrm{Luk}(n)$.

Now let us fix a path $\lambda \in \mathrm{Luk}(n)$, where (consistent to the above) we denote the rise-vector of λ as $\vec{\lambda} := (p_n - 1, \ldots, p_1 - 1)$. If we put

$$\widetilde{p}_m := p_{n+1-m}, \ \widetilde{h}_m := h_{n+1-m}, \ \widetilde{i}_m := i_{n+1-m}, \quad 1 \leq m \leq n,$$

then Equation (6.13) can be re-written in the form

$$\mathrm{term}_{(p_1,\ldots,p_n)} = \langle \xi_{\mathrm{vac}} \, , \, X^*_{\widetilde{p}_1;\widetilde{h}_1} S_{\widetilde{i}_1;\widetilde{h}_1} \cdots X^*_{\widetilde{p}_n;\widetilde{h}_n} S_{\widetilde{i}_n;\widetilde{h}_n} \, \xi_{\mathrm{vac}} \, \rangle,$$

where on the right-hand side we are in the position to invoke Lemma 6.6. The lemma must be used in connection to the path λ and the tuples $\chi_{\mathrm{opp}} = (\widetilde{h}_1, \ldots, \widetilde{h}_n)$, $\omega_{\mathrm{opp}} = (\widetilde{i}_1, \ldots, \widetilde{i}_n)$. If we also denote

$$\widetilde{\pi} := \Phi_{\chi_{\mathrm{opp}}}(\lambda) \quad \text{(output-time partition associated to } \lambda \text{ and } \chi_{\mathrm{opp}}),$$

the application of Lemma 6.6 takes us to:

$$\mathrm{term}_{(p_1,\ldots,p_n)} = \prod_{\widetilde{T} \in \widetilde{\pi}} \widetilde{\gamma}(\, (\omega_{\mathrm{opp}} ; \chi_{\mathrm{opp}}) \mid \widetilde{T}).$$

Finally, we note that when $\widetilde{T}$ runs among the blocks of $\widetilde{\pi}$, the set $(n+1) - \widetilde{T}$ runs among the blocks of the opposite partition $\widetilde{\pi}_{\mathrm{opp}}$. Thus, in view of the relation between γ's and $\widetilde{\gamma}$'s observed in Remark 6.4, we arrive to the formula

$$\mathrm{term}_{(p_1,\ldots,p_n)} = \prod_{T \in (\, \Phi_{\chi_{\mathrm{opp}}}(\lambda) \,)_{\mathrm{opp}}} \gamma(\, (\omega ; \chi) \mid T \,).$$

The overall conclusion of the above discussion is that we have

$$\varphi_{\mathrm{vac}}(C_{i_1;h_1} \cdots C_{i_n;h_n}) = \sum_{\lambda \in \mathrm{Luk}(n)} \ \prod_{T \in (\, \Phi_{\chi_{\mathrm{opp}}}(\lambda) \,)_{\mathrm{opp}}} \gamma(\, (\omega ; \chi) \mid T \,).$$

The only thing left to verify is, then, that the set of partitions

$$\big\{ \, \big(\, \Phi_{\chi_{\mathrm{opp}}}(\lambda) \, \big)_{\mathrm{opp}} \mid \lambda \in \mathrm{Luk}(n) \big\}$$

coincides with $\mathcal{P}^{(\chi)}(n)$. But this is indeed true, since $\{ \Phi_{\chi_{\mathrm{opp}}}(\lambda) \mid \lambda \in \mathrm{Luk}(n) \} = \mathcal{P}^{(\chi_{\mathrm{opp}})}(n)$ (by the definition of $\mathcal{P}^{(\chi_{\mathrm{opp}})}(n)$), and in view of Proposition 4.14. ■

6.9. Proof of Theorem 6.2. We verify the required formula (6.3) by induction on n.

For $n = 1$ we only have to observe that $\kappa_{(\ell)}(A_i) = \gamma(\,(i);(\ell)\,), \quad \forall\, 1 \le i \le d$ (both the above quantities are equal to $\alpha_{(i)}$), and that $\kappa_{(r)}(B_i) = \gamma(\,(i);(r)\,), \quad \forall\, 1 \le i \le d$ (both quantities equal to $\beta_{(i)}$).

Induction step: consider an $n \ge 2$, suppose the equality in (6.3) has already been verified for all bi-words of length $\le n-1$, and let us fix a bi-word $(\omega;\chi)$ of length n, for which we want to verify it as well. Write explicitly $\omega = (i_1, \ldots, i_n)$ and $\chi = (h_1, \ldots, h_n)$, with $1 \le i_1, \ldots, i_n \le d$ and $h_1, \ldots, h_n \in \{\ell, r\}$. The joint moment $\varphi_{\mathrm{vac}}(C_{i_1;h_1} \cdots C_{i_n;h_n})$ can be expressed as a sum over $\mathcal{P}^{(\chi)}(n)$ in two ways: on the one hand we have it written as in Equation (6.11) of Proposition 6.8, and on the other hand we can write it by using the moment$\leftrightarrow$cumulant formula (5.1) which was used to introduce the (ℓ, r)-cumulants in Definition 5.2:

$$\varphi_{\mathrm{vac}}(C_{i_1;h_1} \cdots C_{i_n;h_n}) = \sum_{\pi \in \mathcal{P}^{(\chi)}(n)} \Big(\prod_{V \in \pi} \kappa_{\chi|V}(\,(C_{i_1;h_1}, \ldots, C_{i_n;h_n}) \mid V\,) \Big). \tag{6.14}$$

The induction hypothesis immediately gives us that, for every $\pi \ne 1_n$ in $\mathcal{P}^{(\chi)}(n)$, the term indexed by π in the two summations that were just mentioned (right-hand side of (6.11) and right-hand side of (6.14)) are equal to each other. When we equate these two summations and cancel all the terms indexed by $\pi \ne 1_n$ in $\mathcal{P}^{(\chi)}(n)$, we are left precisely with $\kappa_\chi(C_{i_1;h_1}, \ldots, C_{i_n;h_n}) = \gamma(\omega;\chi)$, as required. ■

Acknowledgements

This research work was started while the authors were participating in the focus program on free probability at the Fields Institute in Toronto, in July 2013. The uplifting atmosphere and the support of the Fields focus program are gratefully acknowledged.

We also express our thanks to the anonymous referee who pointed to us the importance of re-writing the introduction in a way which better shows the motivation of the paper.

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