Magnetic field of a noncircular solenoid

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$$dx_c = \pm \frac{1}{2b} \cos^2 \theta \, d\theta = \pm \frac{1 + \cos 2\theta}{4b} \, d\theta. \tag{11}$$

Integrating, we obtain (only the plus sign is needed)

$$x_c = (1/8b)(2\theta + \sin 2\theta) + c, \tag{12a}$$

and, from (10),

$$y_c = a + (1/8b)(1 - \cos 2\theta). \tag{12b}$$

For the boundary conditions $x_c = 0$, $y_c = r$ for $\theta = 0$, and with the identification $R = 1/8b$, Eqs. (12) yield Gillespie's results

$$x_c = R(2\theta + \sin 2\theta) \tag{13a}$$

$$y_c = r + R(1 - \cos 2\theta). \tag{13b}$$

Thus the center of mass of the rolling object moves on a cycloid generated by a rolling circle of radius R.

Since the Lagrangian (4) for O has the same form as that for a particle sliding on a smooth curve,[5] except that m is replaced by $m(1+\gamma)$, then the curve C is ideally also one which produces the fastest time of descent for the rolling object, i.e., the curve is also the brachistochrone. However, as Gillespie has noted, the friction between O and C required to produce rolling may not be achieved on the steeply sloped portion of C. Indeed, for the brachistochrone curve between two points in the Earth's gravitational field, the rolling object *must* start on the vertical cusp of the cycloid,[3] where the normal force between O and C is zero, and thus there is no friction, and O starts from such a point by sliding, not rolling.

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[4]W. D. MacMillan, *Statics and the Dynamics of a Particle* (Dover, New York, 1958), pp. 225–227.

[5]H. Goldstein, *Classical Mechanics* (Addison-Wesley, Reading, MA, 1950), Chap. 2.

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In a recent note,[1] B. Dasgupta has calculated the magnetic field of a circular solenoid by direct integration over the source current. While that calculation is certainly correct, the following procedure is at once simpler and more general.

Consider an infinitely long cylindrical shell of *arbitrary shape*, such as shown in cross section in Fig. 1. A uniform surface current of density J_s flows "around" the cylinder, that is in the sense of the arrow along curve C. Locate the origin of a cylindrical coordinate system (ρ, ϕ, z) at the point at which the magnetic field is to be calculated. Let us consider first a field point *inside* the cylinder, as shown. A general argument will later suffice to extend our result to points outside the cylinder.

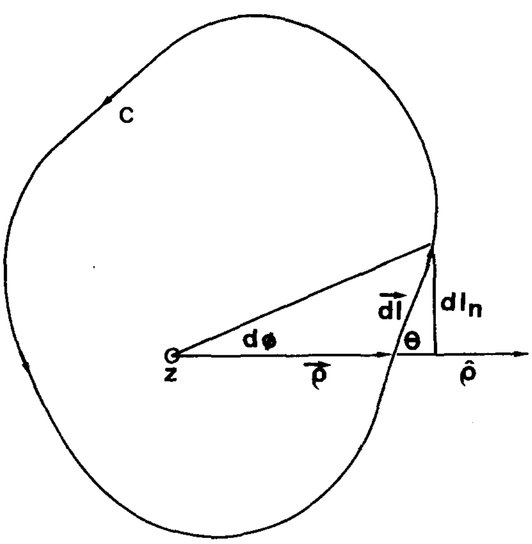

Fig. 1. Cross section of noncircular cylinder.

Let dS be a surface element located at $\mathbf{r} = (\rho, \phi, z)$. We break $\mathbf{r}$ into vertical and horizontal components:

$$\mathbf{r} = z\hat{\mathbf{k}} + \boldsymbol{\rho}, \tag{1}$$

where $\hat{\mathbf{k}}$ is the usual unit vector in the $+z$ direction, up from the page, and $\boldsymbol{\rho}$ is the horizontal vector shown. Since $\hat{\mathbf{k}}$ and $\boldsymbol{\rho}$ are perpendicular,

$$r^2 = z^2 + \rho^2. \tag{2}$$

The current element flowing in dS is $\mathbf{J}_s \, dS$. Let $d\mathbf{l}$ be an element of displacement along the closed curve C, with counter-clockwise positive as usual. The current element can be written

$$\mathbf{J}_s \, dS = J_s \, d\mathbf{l} \, dz. \tag{3}$$

Noting that the vector *from* the current element *to* the field point (origin) is $-\mathbf{r}$, we write for the magnetic field $\mathbf{B}$, from the Biot–Savart law:

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int_S \frac{(\mathbf{J}_s \, dS) \times (-\mathbf{r})}{|-\mathbf{r}|^3}.$$

Substituting from (1), (2), and (3), and writing the surface integral as a double integral over $d\mathbf{l}$ (around the closed path C) and over z (from $-\infty$ to $+\infty$) we get

$$\mathbf{B} = \frac{-\mu_0 J_s}{4\pi} \oint_C \int_{-\infty}^{+\infty} \frac{dz \, d\mathbf{l} \times (z\hat{\mathbf{k}} + \boldsymbol{\rho})}{(z^2 + \rho^2)^{3/2}}. \tag{4}$$

There are two terms arising from the parentheses in the numerator. Considering the first of these, and noting that $d\mathbf{l}$ and $\hat{\mathbf{k}}$ are not functions of z, we must evaluate

$$\oint_C d\mathbf{l} \times \hat{\mathbf{k}} \int_{-\infty}^{+\infty} \frac{z \, dz}{(z^2 + \rho^2)^{3/2}}.$$

In the z integration, the integrand is odd in z, the interval is symmetric, hence the value is zero.

This leaves us with the following expression for **B**, where for convenience we reverse the order of the cross product and absorb the minus sign:

$$\mathbf{B} = \frac{\mu_0 J_s}{4\pi} \oint_C \boldsymbol{\rho} \times d\mathbf{l} \int_{-\infty}^{+\infty} \frac{dz}{(z^2 + \rho^2)^{3/2}}.$$

The z integration is easily done, and gives the value $2/\rho^2$. Using $\hat{\boldsymbol{\rho}} = \boldsymbol{\rho}/\rho$, we obtain

$$\mathbf{B} = \frac{\mu_0 J_s}{2\pi} \oint_C \frac{\hat{\boldsymbol{\rho}} \times d\mathbf{l}}{\rho}.$$

The product $\hat{\boldsymbol{\rho}} \times d\mathbf{l}$ is in the $\hat{\mathbf{k}}$ direction, and of magnitude $dl \sin\theta \equiv dl_n$, where θ is the angle between $\boldsymbol{\rho}$ and $d\mathbf{l}$. From the geometry, dl_n is the component of $d\mathbf{l}$ perpendicular to $\boldsymbol{\rho}$, and dl_n/ρ is the element of angle $d\phi$ subtended by $d\mathbf{l}$ at the origin. Hence,

$$\mathbf{B} = \frac{\mu_0 J_s \hat{\mathbf{k}}}{2\pi} \oint_C d\phi. \tag{5}$$

Now if the field point is *inside* the cylinder, the net change in ϕ in going once around the contour is 2π. Hence

$$\mathbf{B}_{\text{in}} = \mu_0 J_s \hat{\mathbf{k}}.$$

Since this does not depend in any way on the particular location of the field point, we conclude that the field is uniform, and is also independent of the shape of the cylinder.

For a field point located *outside* the cylinder, all equations are identical down to (5). The net change in ϕ in going around an external contour is just 0. Hence

$$\mathbf{B}_{\text{out}} = 0.$$

This result is in agreement with that previously obtained for the particular case of a circular cross section, but is seen to be much more general in application.

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Effect of light polarization on the diffraction pattern of small wires

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I. INTRODUCTION

The subject of light diffraction is treated in many undergraduate textbooks and the examples considered are usually chosen from the theory of diffraction by apertures.[1] This is pedagogically appealing because Huygens' principle may then be used simply to predict the shape of the diffraction pattern. However, light is an electromagnetic wave. In order to describe electromagnetic waves properly, one must normally solve Maxwell's equations, including a description of the nature of the incident wave and the properties of the materials with which it will interact. By being exposed mainly to Huygens' method the student may develop the incorrect perception that diffraction is a phenomenon involving only the geometry of the material, independently of its physical properties like conductivity, absorption coefficient, etc.

According to the theory of diffraction by apertures,[1] the state of polarization of the incident light beam is not required in order to predict the irradiance pattern at large distances. As a result, there has been a tendency not to mention the effects of polarization in classroom discussions, in textbooks,[1] or even in the pedagogical literature,[2] even though such textbooks and articles always mention that the Huygen–Fresnel–Kirchhoff theory is a scalar approximation. One notable exception is due to Fortin,[3] in a context which is different from the present one. In order to show how important the effects of polarization may be in an actual situation, we have designed and performed a diffraction experiment using a very small metallic wire of approximate diameter 30 μm, at a wavelength of 632.8 nm (He–Ne laser). The results indicate quite clearly that the state of polarization of the incident beam is important and we believe that such an experiment could be assigned as a laboratory exercise on diffraction, aimed at illustrating the effect.

II. PHYSICAL ORIGIN OF THE EFFECT

In this section, we consider the scattering of a uniform beam of parallel-polarized light and perpendicular-polarized light which is incident normally upon a perfectly conducting cylinder of radius R. This problem contains the basic ideas required to appreciate the physical origin of the effect presented here. Furthermore, the idealized situation discussed here is mathematically simpler to analyze than a realistic one and so we sketch the former only.

The cylinder axis is made to coincide with the z axis of a cylindrical system (ρ,ϕ,z). By parallel-polarized light, we mean that the electric field of the incident beam is parallel to the wire axis (E_z^o), while the magnetic field is perpendicular to the same axis (B_ρ^o, B_ϕ^o). (The superscript "o" refers to the incident fields; a superscript "s" will be appended to the scattered fields.) In the case of perpendicular-polarized light, we have (E_ρ^o, E_ϕ^o) for the electric field and B_z^o for the magnetic field.

Let $f(\rho,\phi)$ be a function of ρ and ϕ only and expand it in a Fourier series in ϕ,

$$f(\rho,\phi) = \sum_m f_m(\rho) e^{im\phi}, \tag{1}$$

with m an integer $(-\infty \leqslant m \leqslant +\infty)$ and $f_m(\rho)$ a coefficient which depends on ρ. In the present context, $f_m(\rho)$ is the general solution of the standard Bessel equation of order m,

