

A single equation for finite rectangular well energy eigenvalues

B. Cameron Reed

Citation: *American Journal of Physics* **58**, 503 (1990); doi: 10.1119/1.16457

View online: <http://dx.doi.org/10.1119/1.16457>

View Table of Contents: <http://scitation.aip.org/content/aapt/journal/ajp/58/5?ver=pdfcov>

Published by the American Association of Physics Teachers

Articles you may be interested in

[Dispersion equation and eigenvalues for quantum wells using spectral parameter power series](#)

J. Math. Phys. **52**, 043522 (2011); 10.1063/1.3579991

[Comment on "A single equation for finite rectangular well energy eigenvalues," by B. Cameron Reed \[*Am. J. Phys.* **58** \(5\), 503–504 \(1990\)\]](#)

Am. J. Phys. **68**, 1151 (2000); 10.1119/1.1286426

[Energy eigenvalues for an arbitrary potential well with N minima](#)

J. Math. Phys. **32**, 1822 (1991); 10.1063/1.529247

[The energy eigenvalue equation for the linear potential](#)

J. Math. Phys. **19**, 2239 (1978); 10.1063/1.523560

[Energy Eigenvalues in the Finite Square Well](#)

Am. J. Phys. **40**, 922 (1972); 10.1119/1.1986703



American Association of **Physics Teachers**

Explore the **AAPT Career Center** –
access **hundreds of physics education and
other STEM teaching jobs** at two-year and
four-year colleges and universities.

<http://jobs.aapt.org>



mation, $R_H = 26$ m. This means that an archer shooting upward on the slope at a range of 45 m should aim as if the target were at his or her own level at a range of 26 m, since both circumstances require an angle of departure of 2.5 deg. On the other hand, a similar calculation for a shot downward, with the base line still at 60 deg to the horizontal, gives a range of 58 m for the same angle of departure of 2.5 deg. The range is extended more for downhill than for uphill shots because of the significant changes of speed due to gravity.

IV. SUMMARY

The relationships (3) to (6) enable the determination of the shape of a trajectory in air relative to a plane inclined to the horizontal, provided that the angle of departure to the plane is small. Experimental information required is a plot of times of flight (or remaining velocities) against distance traveled for a horizontal range.

For a supersonic bullet, it is a reasonable approximation to neglect the effect of gravity on the speed and to neglect the effect of air resistance on the component of the acceleration normal to the inclined plane. This means that the

times of flight of the bullet for a given range are approximately the same for planes of different slopes. On the other hand, the times of flight for a bullet are not the same for different slopes when the angles of departure are the same. This is because the ranges become longer for steeper slopes.

For an arrow, it is unsatisfactory to neglect completely the effect of gravity on the speed but it is still reasonable to neglect the effect of air resistance on the component of the acceleration normal to the slope.

This treatment applies to ranges of up to about 500 m for supersonic bullets and to about 50 m for arrows.

¹H. R. Kemp, "Trajectories of projectiles in air for small times of flight," *Am. J. Phys.* **55**, 1099-1102 (1987).

²H. R. Kemp, "Projectile paths corrected for recoil and air resistance," *Phys. Educ.* **21**, 19-23 (1986).

³Colonel Townsend Whelen, *Small Arms Design and Ballistics* (Small Arms Technical, Georgetown, 1945), pp. 55-69, 290.

⁴*Encyclopaedia Americana* **3**, 103-107 (1959).

⁵*Encyclopaedia Britannica* **2**, 1003-1004 (1957).

⁶C.L. Farrar and D.W. Leeming, *Military Ballistics—A Basic Manual* (Pergamon, New York, 1983), pp. 73-96.

A single equation for finite rectangular well energy eigenvalues

B. Cameron Reed

Department of Physics, Saint Mary's University, Halifax, Nova Scotia B3H 3C3, Canada

(Received 23 January 1989; accepted for publication 10 April 1989)

The one-dimensional finite rectangular potential well is a standard example via which students are introduced to the concept of even- and odd-parity solutions of the Schrödinger equation. If the well is defined by

$$V(x) = \begin{cases} 0, & -L \leq x \leq L, \\ V_0, & |x| > L, \end{cases} \quad (1)$$

the Schrödinger equation yields

$$\Psi(x) = \begin{cases} Ce^{\beta x} & (x < -L), \\ A \sin(\alpha x) + B \cos(\alpha x) & (-L \leq x \leq L), \\ De^{-\beta x} & (x > L), \end{cases} \quad (2)$$

where

$$\alpha^2 = 2mE/\hbar^2 \quad (3)$$

and

$$\beta^2 = (2m/\hbar^2)(V_0 - E). \quad (4)$$

The continuity of Ψ and its first derivative yields four constraints:

$$-A \sin \xi + B \cos \xi = Ce^{-\eta}, \quad (5)$$

$$\alpha A \cos \xi + \alpha B \sin \xi = \beta Ce^{-\eta}, \quad (6)$$

$$A \sin \xi + B \cos \xi = De^{-\eta}, \quad (7)$$

$$\alpha A \cos \xi - \alpha B \sin \xi = -\beta De^{-\eta}, \quad (8)$$

where $\xi = \alpha L$ and $\eta = \beta L$. In most textbook treatments

these equations are manipulated to give transcendental equations for the bound-state energy eigenvalues as follows. If the sum of Eqs. (5) and (7) is divided by the difference of Eqs. (6) and (8), one finds

$$\xi \tan \xi = \eta. \quad (9)$$

On the other hand, if the difference between Eqs. (5) and (7) is divided by the sum of Eqs. (6) and (8), we have

$$-\xi \cot \xi = \eta. \quad (10)$$

From the definitions of ξ and η there is one further condition:

$$\xi^2 + \eta^2 = 2mV_0L^2/\hbar^2 = \text{const} = K^2. \quad (11)$$

The intersections of Eqs. (9) and (11) give the even-parity solutions ($A = 0$, $C = D$), whereas the intersections of Eqs. (10) and (11) give the odd-parity solutions ($B = 0$, $C = -D$). The solutions are usually found graphically, as demonstrated, for example, in Schiff.¹ The literature abounds with a variety of innovative modifications on Schiff's method,^{2,3} including finding the intersections of semicircles with an Archimedes spiral.⁴ The problem with this approach, however, is that most students find the above series of manipulations to be far from intuitively obvious. Also, in many of the graphical methods, one faces the task of constructing independent graphs for the even and odd solutions. A simpler approach leading to a single equation incorporating all possible solutions is clearly de-

sirable. Indeed, Eqs. (5)–(8) appear at first glance to be four equations in four unknowns, and most students' first intuition is to solve them as such.

The purpose of this note is to show that such an approach does produce a *single* equation, the roots of which yield *all* of the energy eigenvalues. The parities of the corresponding wavefunctions can be shown to follow as a consequence.

The solution proceeds as follows. First, use Eqs. (5) and (6) to express A and B in terms of C and D . With these results, eliminate A and B in Eqs. (7) and (8) to give

$$D = C[\sin(2\xi)(\eta/\xi + \cot \xi) - 1], \quad (12)$$

and

$$D = -C[\cos(2\xi) - (\xi/\eta)\sin(2\xi)], \quad (13)$$

respectively. Equating these results gives an identity in ξ :

$$\sin(2\xi)(\eta/\xi - \xi/\eta + \cot \xi) + \cos(2\xi) = 1. \quad (14)$$

Using Eq. (11) to eliminate η gives, after some manipulation and use of trigonometric identities, a condition on ξ :

$$f(\xi, K) = (K^2 - 2\xi^2)\sin(2\xi) + 2\xi\sqrt{K^2 - \xi^2}\cos(2\xi) = 0. \quad (15)$$

The roots of this equation give the bound-state energy eigenvalues. This result incorporates all possible solutions and, as expressed, possesses the advantage of utilizing no discontinuous trigonometric functions.

The roots of Eq. (15) are most easily found as follows. First, plot the equation and establish values of ξ that bracket the roots. Using these bracketing values as input to a bisection-method program will quickly yield convergence as $f(\xi, K)$ is continuous and possesses no extrema where $f(\xi, K) = 0$. Press *et al.*⁵ discuss root-finding techniques in a very readable way, and give a variety of algorithms and useful practical tips.

As regards the parities of the corresponding wavefunctions, it is easy to show that if either Eq. (12) or (13) is squared and Eq. (15) is used to eliminate $\sin(2\xi)$ or $\cos(2\xi)$, an identity results, namely, $D^2 = C^2$, or $D = \pm C$. If $D = -C$ (odd parity), Eq. (7), on elimination of A via Eq. (5), yields $B = 0$. Similarly, if $D = C$ (even parity), Eq. (5), on elimination of B via Eq. (7), gives $A = 0$.

An example of the use of Eq. (15) is given in Fig. 1, where $f(\xi, K)$ is plotted for the case of an electron in a well of depth 100 eV and width 2 Å ($L = 1$ Å). There are four bound-state solutions, at $\xi = 1.3118, 2.6076, 3.8593,$ and 4.9630 , corresponding to energies of 6.56, 25.91, 56.75, and 93.85 eV, respectively. The vertical dashed line corre-

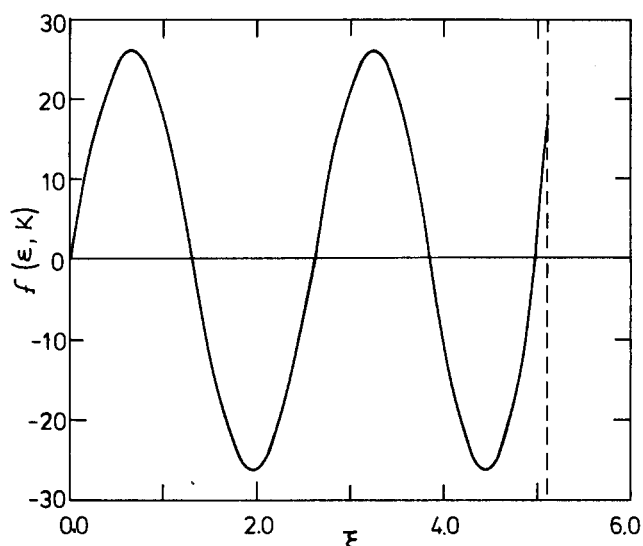


Fig. 1. Equation (15) plotted for the case of an electron in a finite rectangular potential well of depth 100 eV and width 2 Å. The zeros of the curve correspond to the permissible energy eigenvalues. The dashed line indicates the top of the well.

sponds to the top of the well. It is straightforward to verify numerically that the lowest energy solution is of even parity and that the solutions alternate parity.

The author's experience is that students find it easier to digest the present method than that usually presented (and often only vaguely explained) in the standard texts. The generality and relative simplicity of Eq. (15), in combination with the capabilities of programmable calculators and personal computers, make it possible to discuss meaningful numerical examples.⁶

¹ Leonard I. Schiff, *Quantum Mechanics* (McGraw-Hill, New York, 1968), 3rd ed., pp. 41–42.

² Paul H. Pitkanen, "Rectangular potential well problem in quantum mechanics," *Am. J. Phys.* **23**, 111–113 (1955).

³ James J. Murphy, "Comment on 'Bound state energies of a particle in a finite square well,'" *Am. J. Phys.* **39**, 976 (1971).

⁴ W. C. Elmore, "Eigenvalues of the square-well problem," *Am. J. Phys.* **39**, 976–977 (1971).

⁵ William H. Press, Brian P. Flannery, Saul A. Teukolsky, and William T. Vetterling, *Numerical Recipes* (Cambridge U. P., New York, 1986), Chap. 9.

⁶ See, along these lines, R. D. Murphy and J. M. Phillips, "Bound-state eigenvalues of the square-well potential," *Am. J. Phys.* **44**, 574–576 (1976).