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The structure of superqubit states

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ABSTRACT

Superqubits provide a supersymmetric generalisation of the conventional qubit in quantum information theory. After a review of their current status, we address the problem of generating entangled states. We introduce the global unitary supergroup $\mathrm{UOSp}((3^n+1)/2|(3^n-1)/2)$ for an n-superqubit system, which contains as a subgroup the local unitary supergroup $[\mathrm{UOSp}(2|1)]^n$. While for $4 > n > 1$ the bosonic subgroup in $\mathrm{UOSp}((3^n+1)/2|(3^n-1)/2)$ does not contain the standard global unitary group $\mathrm{SU}(2^n)$, it does have an $\mathrm{USp}(2^n) \subset \mathrm{SU}(2^n)$ subgroup which acts transitively on the n-qubit subspace, as required for consistency with the conventional multi-qubit framework. For two superqubits the $\mathrm{UOSp}(5|4)$ action is used to generate entangled states from the "bosonic" separable state $|00\rangle$.

Contents

1 Introduction

Superqubits [1–3] belong to a $(2|1)$-dimensional complex super-Hilbert space [4–6]. In this sense they constitute the minimal supersymmetric generalisation of the conventional qubit. The use of super-Hilbert spaces implies a non-trivial departure from standard quantum theory; the term supersymmetry in this context should not be confused with its more familiar usage in the field of high energy physics, which operates exclusively in the realm of standard quantum theory. While the infinitesimal symmetries and representations relevant to superqubits appear in a variety of physically motivated contexts [7–16], it is not clear what relation they have to superqubits themselves, especially given the use of super-Hilbert space.

Here we review superqubits and develop further the basic formalism, paying particular attention to the issue of generating entangled states using global super-unitary transformations. In section 2 we briefly recall the essential features of conventional qubits. In section 3 we review in some detail Grassmann numbers, supermatrices, supergroups and Lie superalgebras. Having provided the necessary background, in section 4 we review the superqubit formalism and introduce for the first time the super-unitary group acting globally on multi-superqubit states. It shown that this group generates superentangled states.

Recall, a collection of n distinct isolated qubits transforms under the group of local unitaries $[\mathrm{SU}(2)]^n$. The local unitary transformations, by construction, cannot generate entanglement; a separable n-qubit state will remain separable under all $[\mathrm{SU}(2)]^n$ operations. To generate an arbitrary state, entangled or otherwise, from any given initial state one conventionally employs the group of global unitaries $\mathrm{SU}(2^n)$, which acts transitively on the n-qubit state space.

Similarly, a collection of n distinct isolated superqubits transforms under the local unitary orthosymplectic supergroup $[\mathrm{UOSp}(2|1)]^n$, which contains as its bosonic subgroup the conventional local unitaries $[\mathrm{SU}(2)]^n$. Again, being local, this set of transformations is insufficient to generate super-entanglement [1]. With this issue in mind we introduce here the n-superqubit global unitary supergroup given by $\mathrm{UOSp}((3^n+1)/2|(3^n-1)/2)$, the super-analog of $\mathrm{SU}(2^n)$, which is uniquely determined by the single superqubit limit.

At first sight this appears to present a conundrum. For consistency the bosonic subgroup of $\mathrm{UOSp}((3^n+1)/2|(3^n-1)/2)$ is required to act transitively on the subspace of regular qubits sitting inside the super-Hilbert space. However, for $4 > n > 1$ the standard group of global unitaries $\mathrm{SU}(2^n)$ is *not* contained in the bosonic subgroup of $\mathrm{UOSp}((3^n+1)/2|(3^n-1)/2)$. Its absence, it would seem, obstructs the expected consistent reduction to standard qubits. All is not lost however, since a proper subgroup $\mathrm{USp}(2^n) \subset \mathrm{SU}(2^n)$ is sufficient to generate any state from any initial state. This smaller group is indeed always contained in the bosonic subgroup of $\mathrm{UOSp}((3^n+1)/2|(3^n-1)/2)$ and, as we shall explain, acts transitively on the subspace of standard n-qubit states. This is related to the observation made in the context of quantum control [17] that the global unitary group carries a degree of redundancy and a $\mathrm{USp}(2^n)$ subgroup is always sufficient to generate all states, entangled or otherwise.

2 Qubits and global unitary groups

Before turning the case of superqubits let us review the familiar case of regular qubits. The complex projective n-qubit state space

$$\mathbb{P}(\mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2) \cong \mathbb{CP}^{2^n-1} \cong \mathrm{SU}(2^n)/\,\mathrm{U}(2^n-1) \cong S^{2^{n+1}-1}/\,\mathrm{U}(1) \tag{2.1}$$

is acted on transitively and effectively by $\mathrm{SU}(2^n)$. As observed in the context of quantum control [17] there is a proper subgroup $\mathrm{USp}(2^n) \subset \mathrm{SU}(2^n)$, which also acts transitively on the state space $\mathbb{CP}^{2^n-1}$, since

$$\mathrm{USp}(2^n)/[\mathrm{U}(1) \times \mathrm{USp}(2^n-2)] \cong \mathbb{CP}^{2^n-1} \tag{2.2}$$

via the identification $\mathbb{H}^m \cong \mathbb{C}^{2m}$.

Neglecting the U(1) quotient group and so distinguishing normalised states with distinct phase, yields $S^{2^{n+1}-1}$. The possible transitive actions on spheres were classified in [18,19] and are summarised in Table 1 (see also [20]). Table 1 presents a number of possible $\mathrm{U}(2^n)$ subgroups acting transitively on spheres, but it is only $\mathrm{USp}(2^n)$ that is relevant to the case of finite-dimensional quantum systems described by finite-dimensional bilinear models [17].

The conclusion of these observations is that any final n-qubit state, entangled or otherwise, may be obtained from any initial n-qubit state using only the $\mathrm{USp}(2^n)$ subgroup of the familiar unitary group $\mathrm{U}(2^n)$.

Note, while $\mathrm{USp}(2^n) \subset \mathrm{SU}(2^n)$ acts transitively on $\mathbb{CP}^{2^n-1}$, the group of local unitaries appropriate to the case of isolated qubits,

$$\mathrm{SU}_1(2) \times \ldots \times \mathrm{SU}_n(2) \subset \mathrm{SU}(2^n), \tag{2.3}$$

is not a subgroup of $\mathrm{USp}(2^n)$. Although there is an $[\mathrm{SU}(2)]^n$ subgroup in $\mathrm{USp}(2^n)$ it cannot necessarily be identified with the local unitaries for $n > 1$. This is most easily seen at $n = 2$, for which we have the following branching

$$\begin{array}{ccccc} \mathrm{SU}(4) & \supset & \mathrm{USp}(4) & \supset & \mathrm{SU}(2) \times \mathrm{SU}(2) \\ \mathbf{4} & \rightarrow & \mathbf{4} & \rightarrow & (\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2}) \end{array} \tag{2.4}$$

The $\mathrm{SU}(2)\times\mathrm{SU}(2)$ subgroup in $\mathrm{USp}(4)$ is unique (up to conjugation) and therefore cannot be identified with the local unitaries $\mathrm{SU}_A(2) \times \mathrm{SU}_B(2) \subset \mathrm{SU}(4)$ since

$$\begin{array}{ccc} \mathrm{SU}(4) & \supset & \mathrm{SU}_A(2) \times \mathrm{SU}_B(2) \\ \mathbf{4} & \rightarrow & (\mathbf{2},\mathbf{2}) \end{array} \tag{2.5}$$

Isometry group G	Sphere	Stabiliser group K	m
$\mathrm{SO}(n)$	S^{n-1}	$\mathrm{SO}(n-1)$	0
$\mathrm{U}(n)$	S^{2n-1}	$\mathrm{U}(n-1)$	1
$\mathrm{SU}(n)$		$\mathrm{SU}(n-1)$	1
$\mathrm{USp}(n) \times \mathrm{USp}(2)$	S^{4n-1}	$\mathrm{USp}(n-1) \times \mathrm{USp}(2)$	1
$\mathrm{USp}(n) \times \mathrm{U}(1)$		$\mathrm{USp}(n-1) \times \mathrm{U}(1)$	2
$\mathrm{USp}(n)$		$\mathrm{USp}(n-1)$	6
G_2	S^6	$\mathrm{SU}(3)$	0
$\mathrm{Spin}(7)$	S^7	G_2	0
$\mathrm{Spin}(9)$	S^{15}	$\mathrm{Spin}(7)$	1

Table 1: Transitive actions on spheres. Here K denotes the isotropy subgroup and m indicates the dimension of the space of G-invariant Riemannian metrics up to homotheties.

3 Supergroups

3.1 Grassmann numbers

Grassmann numbers are elements of the Grassmann algebra Λ_n over $\mathbb{C}$ (or $\mathbb{R}$ with analogous definitions), which is generated by n mutually anticommuting elements $\{\theta^i\}_{i=1}^n$.

Any Grassmann number z may be decomposed into "body" $z_B \in \mathbb{C}$ and "soul" z_S viz.

$$z = z_B + z_S, \quad \text{where} \quad z_S = \sum_{k=1}^{\infty} \frac{1}{k!} c_{a_1\cdots a_k} \theta^{a_1} \cdots \theta^{a_k}, \tag{3.1}$$

and $c_{a_1\cdots a_k} \in \mathbb{C}$ are totally antisymmetric. For finite dimension n the sum terminates at $k = 2^n$ and the soul is nilpotent $z_S^{n+1} = 0$. One can take the formal limit $n \rightarrow \infty$, in which case elements of the algebra are often refereed to as supernumbers.

One may also decompose z into even and odd parts $u \in \Lambda_n^0$ and $v \in \Lambda_n^1$

$$\begin{aligned} u &= z_{\mathcal{B}} + \sum_{k=1}^{\infty} \frac{1}{(2k)!} c_{a_1\cdots a_{2k}} \theta^{a_1} \cdots \theta^{a_{2k}} \\ v &= \sum_{k=0}^{\infty} \frac{1}{(2k+1)!} c_{a_1\cdots a_{2k+1}} \theta^{a_1} \cdots \theta^{a_{2k+1}}, \end{aligned} \tag{3.2}$$

where $\Lambda_n = \Lambda_n^0 \oplus \Lambda_n^1$. For a Grassmann algebra over the complexes we also use $\mathbb{C}_c$ and $\mathbb{C}_a$ for the even commuting and odd anticommuting parts.

One defines the *grade of a Grassmann number* as

$$\deg x := \begin{cases} 0 & x \in \Lambda_n^0 \\ 1 & x \in \Lambda_n^1, \end{cases} \tag{3.3}$$

where the grades 0 and 1 are referred to as even and odd, respectively. Note, $xy = (-)^{xy} yx$ for $x, y \in \Lambda_n^i$. Here we have introduced the shorthand notation, $\deg \alpha \to \alpha$, for any $\deg \alpha$ appearing in the exponent of $(-)$.

The superstar $^\# : \Lambda_n^i \to \Lambda_n^i$ is defined to satisfy,

$$(x\theta_i)^\# = x^* \theta_i^\#, \quad \theta_i^{\#\#} = -\theta_i, \quad (\theta_i \theta_j)^\# = \theta_i^\# \theta_j^\#, \tag{3.4}$$

where $x \in \mathbb{C}$ and * is ordinary complex conjugation [21, 22]. Hence,

$$\alpha^{\#\#} = (-)^\alpha \alpha \tag{3.5}$$

for pure even/odd Grassmann α. The impure case follows by linearity.

3.2 Supermatrices

A $(p|q) \times (r|s)$ supermatrix is a $(p+q) \times (r+s)$-dimensional block partitioned matrix

$$M = \begin{array}{c} \\ p \\ q \end{array} \overset{\begin{array}{cc} r & s \end{array}}{\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]} \tag{3.6}$$

with entries in a Grassmann algebra, where supermatrix multiplication is defined as for ordinary matrices. Note, the special cases $r = 1, s = 0$ or $p = 1, q = 0$ correspond to row and column supervectors. For notational convenience we will denote the $(p|q)$-dimensional supervector space over a complex (real) Grassmann algebra by $\mathbb{C}^{p|q}$ $(\mathbb{R}^{p|q})$.

A supermatrix has a definite grade $\deg M = 0, 1$ if Grassmann entries in the A and D blocks are grade $\deg M$, and those in the B and C blocks are grade $\deg M + 1 \mod 2$. The supertranspose M^{st} of a supermatrix M with $\deg M$ is defined componentwise as

$$M^{st}{}_{X_1 X_2} := (-)^{(X_1+M)(X_1+X_2)} M_{X_2 X_1}, \tag{3.7}$$

where $X_1 = 1, \ldots p|p+1, \ldots p+q$ and $X_2 = 1, \ldots r|r+1, \ldots r+s$. Note, we have assigned supermatrix indices a grade in the obvious manner and addition in the exponent of $(-)$ is always mod 2.

In block matrix form

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{st} = \begin{bmatrix} A^t & (-)^M C^t \\ (-)^{M+1} B^t & D^t \end{bmatrix}, \tag{3.8}$$

so for column (row) vectors V (W) we have,

$$V^{st} = \begin{bmatrix} x \\ y \end{bmatrix}^{st} = \begin{bmatrix} x^t & (-)^V y^t \end{bmatrix}, \quad W^{st} = \begin{bmatrix} w & z \end{bmatrix}^{st} = \begin{bmatrix} w^t \\ (-)^{W+1} z^t \end{bmatrix}. \tag{3.9}$$

The supertranspose of an inhomogeneous grade supermatrix is defined by linearity.

The supertranspose satisfies

$$\begin{aligned} M^{st\,st}{}_{X_1 X_2} &= (-)^{(X_1+X_2)} M_{X_1 X_2}, \\ M^{st\,st\,st}{}_{X_1 X_2} &= (-)^{(X_2+M)(X_1+X_2)} M_{X_2 X_1}, \\ M^{st\,st\,st\,st}{}_{X_1 X_2} &= M_{X_1 X_2}, \end{aligned} \tag{3.10}$$

and

$$(MN)^{st} = (-)^{MN} N^{st} M^{st}. \tag{3.11}$$

The superadjoint ‡ of a supermatrix is defined as

$$M^\ddagger := M^{\# st}, \tag{3.12}$$

and satisfies

$$M^{\ddagger\ddagger} = (-)^M M, \quad (MN)^\ddagger = (-)^{MN} N^\ddagger M^\ddagger. \tag{3.13}$$

Note, the preservation of anti-super-Hermiticity, $M^\ddagger = -M$, under scalar multiplication by Grassmann numbers necessitates the left/right multiplication rules [23],

$$\begin{aligned}(\alpha M)_{X_1X_2} &= (-)^{X_1\alpha}\alpha M_{X_1X_2},\\ (M\alpha)_{X_1X_2} &= (-)^{X_2\alpha} M_{X_1X_2}\alpha,\end{aligned} \tag{3.14}$$

or in block matrix form

$$\alpha\begin{bmatrix}A & B\\ C & D\end{bmatrix} = \begin{bmatrix}\alpha A & \alpha B\\ (-)^\alpha\alpha C & (-)^\alpha\alpha D\end{bmatrix}, \qquad \begin{bmatrix}A & B\\ C & D\end{bmatrix}\alpha = \begin{bmatrix}A\alpha & (-)^\alpha B\alpha\\ C\alpha & (-)^\alpha D\alpha\end{bmatrix}. \tag{3.15}$$

The direct sum and super tensor product are unchanged from their ordinary versions up to the application of the sign rule in the commutatively isomorphism,

$$M\otimes N \mapsto (-)^{MN} N\otimes M, \tag{3.16}$$

and multiplication rule

$$(M_1\otimes N_1)(M_2\otimes N_2) = (-)^{N_1M_2} M_1M_2\otimes N_1N_2. \tag{3.17}$$

3.3 Orthosymplectic supergroups

We will need to consider two isomorphic sets of Lie supergroups,

$$\mathrm{OSp}(2p+1|2q) \quad \text{and} \quad \mathrm{OSp}(2q|2p+1), \tag{3.18}$$

which are special cases of $\mathrm{OSp}(r|2q)$ and $\mathrm{OSp}(2q|r)$, respectively. See, for example, [1] and the references therein.

As supermatrix groups they are defined as,

$$\begin{aligned}\mathrm{OSp}(2p+1|2q) &:= \left\{X\in \mathrm{GL}(2p+1|2q) \,|\, X^{st}(\eta\oplus\Omega)X = \eta\oplus\Omega\right\},\\ \mathrm{OSp}(2q|2p+1) &:= \left\{X\in \mathrm{GL}(2q|2p+1) \,|\, X^{st}(\Omega\oplus\eta)X = \Omega\oplus\eta\right\},\end{aligned} \tag{3.19}$$

where η and Ω are the symmetric and sympletic bilinear forms of $\mathrm{SO}(2p+1,\mathbb{C})$ and $\mathrm{Sp}(2q,\mathbb{C})$, respectively, and $\mathrm{GL}(r|s)$ denotes the supergroup of invertible $(r|s)\times(r|s)$ even supermatrices. In the following we will only discuss $\mathrm{OSp}(2p+1|2q)$ as the structure of $\mathrm{OSp}(2q|2p+1)$ trivially follows; one simply sends $\delta_{X_1X_4}\delta_{X_2X_3}$ to $\delta_{X_1X_3}\delta_{X_2X_4}$ in the definition of U given in (3.20) below.

The $\mathfrak{osp}(2p+1|2q)$ superalgebra in the defining representation can be constructed using the supermatrices

$$(U_{X_1X_2})_{X_3X_4} := \delta_{X_1X_4}\delta_{X_2X_3}, \quad \text{and} \quad G := \left[\begin{array}{c|c}\eta & 0\\ \hline 0 & \Omega\end{array}\right]. \tag{3.20}$$

Here the indices X_i range from 1 to $2p+1+2q$ and are partitioned as $X_i = (\mu, a)$ with μ ranging from 1 to $2p+1$, and a taking on the remaining $2q$ values.

The generators $T\in\mathfrak{osp}(2p+1|2q)$ are then given by,

$$T_{X_1X_2} = 2G_{[\![X_1|X_3}U_{X_3|X_2]\!]}, \tag{3.21}$$

where T has array grade zero and we have introduced the graded symmetrization of superarrays,

$$M_{X_1\cdots[\![X_i|\cdots|X_j]\!]\cdots X_k} := \tfrac{1}{2}[M_{X_1\cdots X_i\cdots X_j\cdots X_k} + (-)^{(X_i+1)(X_j+1)}M_{X_1\cdots X_j\cdots X_i\cdots X_k}]. \tag{3.22}$$

Explicitly,

$$
\begin{aligned}
T_{\mu\nu} &= G_{\mu\lambda}U_{\lambda\nu} - G_{\nu\lambda}U_{\lambda\mu}, \\
T_{ab} &= G_{ac}U_{cb} + G_{bc}U_{ca}, \\
T_{\mu b} &= G_{\mu\lambda}U_{\lambda b} + G_{bc}U_{c\mu}.
\end{aligned} \tag{3.23}
$$

Clearly T has symmetry properties $T_{X_1X_2} = T_{[\![X_1X_2]\!]}$. The $2q(2q+1)/2$ elements $T_{a_1a_2}$ generate $\mathfrak{sp}(2q)$, the $(2p+1)(2p)/2$ elements $T_{\mu_1\mu_2}$ generate $\mathfrak{so}(2p+1)$, and both are even (bosonic), while the $(2p+1)2q$ generators $T_{\mu a}$ are odd (fermionic). These supermatrices yield the $\mathfrak{osp}(2p+1|2q)$ superbrackets

$$
[\![T_{X_1X_2}, T_{X_3X_4}]\!] := 4G_{[\![X_1[\![X_3}T_{X_2]\!]X_4]\!]}, \tag{3.24}
$$

where the supersymmetrization on the right-hand side is over pairs X_1X_2, X_3X_4 as on the left-hand side and we have defined the superbracket

$$
[\![M_{X_1X_2}, N_{X_3X_4}]\!] := M_{X_1X_2}N_{X_3X_4} - (-)^{(X_1+X_2)(X_3+X_4)}N_{X_3X_4}M_{X_1X_2}. \tag{3.25}
$$

The action of the generators on $\mathbb{C}^{(2p+1|2q)}$, which constitutes a left $\mathfrak{osp}(2p+1|2q)$-supermodule, is given by

$$
(T_{X_1X_2})_{X_3X_4}a_{X_4} \equiv (T_{X_1X_2}a)_{X_3} = 2G_{[\![X_1|X_3}a_{X_2]\!]}, \quad a \in \mathbb{C}^{2p+1|2q}. \tag{3.26}
$$

This action may be generalized to an N-fold super tensor product of $(2p+1|2q)$ supervectors by labeling the indices with integers $k = 1, 2, \ldots, N$

$$
(T_{X_kY_k}a)_{Z_1\cdots Z_k\cdots Z_N} = (-)^{(X_k+Y_k)\sum_{i=1}^{k-1}|Z_i|}2G_{[\![X_k|Z_k}a_{Z_1\cdots|Y_k]\!]\cdots Z_N}. \tag{3.27}
$$

The Grassmann envelop or left supermodule $\mathfrak{osp}(2p+1|2q;\mathfrak{G})$ [22, 24] is given by the set of even supermatrices

$$
X = \xi_{X_1X_2}T_{X_1X_2} \tag{3.28}
$$

where the $\xi_{X_1X_2}$ are even or odd complex Grassmann numbers if $\deg(X_1) + \deg(X_2) = 0$ or 1, respectively. The identity connected component of the supergroup $\mathrm{OSp}(2p+1|2q)$ is given by the exponential map of $\mathfrak{osp}(2p+1|2q;\mathfrak{G})$.

3.4 Unitary orthosymplectic supergroups

As a supermatrix group the "real form" $\mathrm{UOSp}(2p+1|2q)$ is defined as,

$$
\mathrm{UOSp}(2p+1|2q) := \Big\{X \in \mathrm{OSp}(2p+1|2q) \,|\, X^\ddagger = X^{-1}, \quad \mathrm{Ber}(X) = 1\Big\}, \tag{3.29}
$$

where

$$
\mathrm{Ber}\, M := \det(A - BD^{-1}C)/\det(D) = \det(A)/\det(D - CA^{-1}B) \tag{3.30}
$$

is the Berezinian.

The corresponding Lie algebra is given by,

$$
\mathfrak{uosp}(2p+1|2q;\mathfrak{G}) := \{X \in \mathfrak{osp}(2p+1|2q;\mathfrak{G}) | X^\ddagger = -X\}. \tag{3.31}
$$

Under supertransposition the $\mathfrak{osp}(2p+1|2q)$ generators obey

$$
T_{X_1X_2}{}^{st} = (-)^{X_1+X_2}G_{X_1X_1'}G_{X_2X_2'}T_{X_1'X_2'} \tag{3.32}
$$

or in blocks

$$
T_{\mu\nu}{}^{st} = -\eta_{\mu\mu'}\eta_{\nu\nu'}T_{\mu'\nu'}, \qquad T_{ab}{}^{st} = -\Omega_{aa'}\Omega_{bb'}T_{a'b'}, \qquad T_{\mu b}{}^{st} = \eta_{\mu\mu'}\Omega_{bb'}T_{\mu'b'}. \tag{3.33}
$$

Note, here we are taking the supertranspose of the supermatrices $T_{X_1X_2}$ defined by (3.21); the indices here label an $X_1 \times X_2$ array of supermatrices, not elements of a supermatrix. Using (3.32) even grade elements $X \in \mathfrak{uosp}(2p+1|2q;\mathfrak{G})$ can be written,

$$
X = (\alpha_{\mu\nu} + \eta_{\mu\mu'}\eta_{\nu\nu'}\alpha^{\#}_{\mu'\nu'})T_{\mu\nu} + (\beta_{ab} + \Omega_{aa'}\Omega_{bb'}\beta^{\#}_{a'b'})T_{ab}, \tag{3.34}
$$

where α, β are commuting Grassmann numbers. The odd grade elements $X \in \mathfrak{uosp}(2p+1|2q;\mathfrak{G})$ are give by,

$$X = (\tau_{\mu b} + \eta_{\mu\mu'}\Omega_{bb'}\tau^{\#}_{\mu' b'})T_{\mu b}, \tag{3.35}$$

where τ are anticommuting Grassmann numbers. Hence, a generic element $X \in \mathfrak{uosp}(2p+1|2q;\mathfrak{G})$ can be written,

$$X = (\alpha_{\mu\nu} + \eta_{\mu\mu'}\eta_{\nu\nu'}\alpha^{\#}_{\mu'\nu'})T_{\mu\nu} + (\beta_{ab} + \Omega_{aa'}\Omega_{bb'}\beta^{\#}_{a'b'})T_{ab} + (\tau_{\mu b} + \eta_{\mu\mu'}\Omega_{bb'}\tau^{\#}_{\mu' b'})T_{\mu b}. \tag{3.36}$$

4 Superqubits and global unitary supergroups

The n-superqubit states are given by elements

$$|\psi\rangle = a_{X_1\dots X_2}|X_1\dots X_2\rangle, \qquad X_i = 0, 1, \bullet \tag{4.1}$$

of the n-fold tensor product super Hilbert space

$$\mathbb{C}^{2|1}_1 \otimes \dots \otimes \mathbb{C}^{2|1}_n, \tag{4.2}$$

which constitutes the fundamental representation of the super local operations group $[\mathrm{UOSp}(2|1)]^{\otimes n}$.

The supergroup of global unitary operations is given by

$$\mathrm{UOSp}(\tfrac{3^n+1}{2}|\tfrac{3^n-1}{2}) \tag{4.3}$$

acting on the 3^n-dimensional graded vector space

$$\mathbb{C}^{\frac{3^n+1}{2}|\frac{3^n-1}{2}} \cong \mathbb{C}^{2|1}_1 \otimes \dots \otimes \mathbb{C}^{2|1}_n. \tag{4.4}$$

Note, here we have reordered the alternating even/odd grades inherited from the tensor product into to the standard $\frac{3^n+1}{2}|\frac{3^n-1}{2}$ convention.

The global supergroup (4.3) has body,

$$\mathrm{USp}(\tfrac{3^n+1}{2}) \times \mathrm{SO}(\tfrac{3^n-1}{2}), \qquad \mathrm{SO}(\tfrac{3^n+1}{2}) \times \mathrm{USp}(\tfrac{3^n-1}{2}), \tag{4.5}$$

for $n = 2m+1$ and $n = 2m$, respectively. Under the bosonic subgroup (4.1) transforms in the defining representation

$$V_n \otimes \mathbf{1} \oplus \mathbf{1} \otimes W_n, \qquad W_n \otimes \mathbf{1} \oplus \mathbf{1} \otimes V_n, \tag{4.6}$$

for $n = 2m+1$ and $n = 2m$, respectively, where V_n and W_n denote the defining vector and symplectic vector representations of SO and USp. Under the $[\mathrm{SU}(2)]^n$ body of $[\mathrm{UOSp}(2|1)]^n \subset \mathrm{UOSp}(\frac{3^n+1}{2}|\frac{3^n-1}{2})$ the n-superqubit representation (4.6) branches to

$$\bigoplus_{p=1}^{n}\left[\bigoplus_{\text{distinct permutations}}^{{}^nC_p}(\overbrace{\mathbf{2},\mathbf{2},\dots,\mathbf{2}}^{p},\underbrace{\mathbf{1},\mathbf{1},\dots,\mathbf{1}}_{n-p})\right]. \tag{4.7}$$

The even (odd) graded subspaces are given by $n-p$ even (odd).

4.1 Two superqubits: $\mathrm{UOSp}(5|4) \supset \mathrm{UOSp}_A(2|1) \times \mathrm{UOSp}_B(2|1)$

Two superqubit states

$$|\psi\rangle = a_{XY}|XY\rangle = a_{AB}|AB\rangle + a_{A\bullet}|A\bullet\rangle + a_{\bullet B}|\bullet B\rangle + a_{\bullet\bullet}|\bullet\bullet\rangle \tag{4.8}$$

transforms as the

$$(\mathbf{5},\mathbf{1}) \oplus (\mathbf{1},\mathbf{4}) \tag{4.9}$$

of $\mathrm{SO}(5) \times \mathrm{USp}(4)$, where the even subspace spanned by $\{|AB\rangle, |\bullet\bullet\rangle\}$ constitutes the $(\mathbf{5},\mathbf{1})$ and the odd subspace spanned by $\{|A\bullet\rangle, |\bullet B\rangle\}$ constitutes the $(\mathbf{1},\mathbf{4})$.

There is a unique SU(2) × SU(2) in SO(5) ≅ UOSp(4) under which

$$\mathbf{5} \rightarrow (\mathbf{1},\mathbf{1}) \oplus (\mathbf{2},\mathbf{2}), \qquad \mathbf{4} \rightarrow (\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2}). \tag{4.10}$$

Hence, there is a diagonal $\mathrm{SU}_A \times \mathrm{SU}_B$ in $[\mathrm{SU}(2)]^4 \subset \mathrm{SO}(5) \times \mathrm{UOSp}(4)$ under which

$$(\mathbf{5},\mathbf{1}) \oplus (\mathbf{1},\mathbf{4}) \rightarrow (\mathbf{2},\mathbf{2}) \oplus (\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2}) \oplus (\mathbf{1},\mathbf{1}), \tag{4.11}$$

where the summands are spanned by $\{|AB\rangle\}, \{|A\bullet\rangle\}, \{|\bullet B\rangle\}$ and $\{|\bullet\bullet\rangle\}$, respectively.

Letting

$$\eta = \begin{bmatrix} \mathbb{1} & 0 & 0 \\ 0 & \mathbb{1} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \Omega = \begin{bmatrix} 0 & \mathbb{1} \\ -\mathbb{1} & 0 \end{bmatrix}, \tag{4.12}$$

a general element $X \in \mathfrak{uosp}(5|4; \mathfrak{G})$ may be written,

$$X = \left[\begin{array}{ccccc|cccc}
0 & -\alpha^{\#}_{12} - \alpha_{12} & -\alpha^{\#}_{13} - \alpha_{13} & -\alpha^{\#}_{14} - \alpha_{14} & -\alpha^{\#}_{15} - \alpha_{15} & \tau^{\#}_{11} - \tau_{13} & \tau^{\#}_{12} - \tau_{14} & \tau^{\#}_{13} + \tau_{11} & \tau^{\#}_{14} + \tau_{12} \\
\alpha_{12} + \alpha^{\#}_{12} & 0 & -\alpha^{\#}_{23} - \alpha_{23} & -\alpha^{\#}_{24} - \alpha_{24} & -\alpha^{\#}_{25} - \alpha_{25} & \tau^{\#}_{21} - \tau_{23} & \tau^{\#}_{22} - \tau_{24} & \tau^{\#}_{23} + \tau_{21} & \tau^{\#}_{24} + \tau_{22} \\
\alpha_{13} + \alpha^{\#}_{13} & \alpha_{23} + \alpha^{\#}_{23} & 0 & -\alpha^{\#}_{34} - \alpha_{34} & -\alpha^{\#}_{35} - \alpha_{35} & \tau^{\#}_{31} - \tau_{33} & \tau^{\#}_{32} - \tau_{34} & \tau^{\#}_{33} + \tau_{31} & \tau^{\#}_{34} + \tau_{32} \\
\alpha_{14} + \alpha^{\#}_{14} & \alpha_{24} + \alpha^{\#}_{24} & \alpha_{34} + \alpha^{\#}_{34} & 0 & -\alpha^{\#}_{45} - \alpha_{45} & \tau^{\#}_{41} - \tau_{43} & \tau^{\#}_{42} - \tau_{44} & \tau^{\#}_{43} + \tau_{41} & \tau^{\#}_{44} + \tau_{42} \\
\alpha_{15} + \alpha^{\#}_{15} & \alpha_{25} + \alpha^{\#}_{25} & \alpha_{35} + \alpha^{\#}_{35} & \alpha_{45} + \alpha^{\#}_{45} & 0 & \tau^{\#}_{51} - \tau_{53} & \tau^{\#}_{52} - \tau_{54} & \tau^{\#}_{53} + \tau_{51} & \tau^{\#}_{54} + \tau_{52} \\
\hline
-\tau^{\#}_{13} - \tau_{11} & -\tau^{\#}_{23} - \tau_{21} & -\tau^{\#}_{33} - \tau_{31} & -\tau^{\#}_{43} - \tau_{41} & -\tau^{\#}_{53} - \tau_{51} & \beta^{\#}_{13} - \beta_{13} & \beta^{\#}_{23} - \beta_{14} & \beta_{11} + \beta^{\#}_{33} & \beta_{12} + \beta^{\#}_{34} \\
-\tau^{\#}_{14} - \tau_{12} & -\tau^{\#}_{24} - \tau_{22} & -\tau^{\#}_{34} - \tau_{32} & -\tau^{\#}_{44} - \tau_{42} & -\tau^{\#}_{54} - \tau_{52} & \beta^{\#}_{14} - \beta_{23} & \beta^{\#}_{24} - \beta_{24} & \beta_{12} + \beta^{\#}_{34} & \beta_{22} + \beta^{\#}_{44} \\
-\tau_{13} + \tau^{\#}_{11} & -\tau_{23} + \tau^{\#}_{21} & -\tau_{33} + \tau^{\#}_{31} & -\tau_{43} + \tau^{\#}_{41} & -\tau_{53} + \tau^{\#}_{51} & -\beta^{\#}_{11} - \beta_{33} & -\beta^{\#}_{12} - \beta_{34} & \beta_{13} - \beta^{\#}_{13} & \beta_{23} - \beta^{\#}_{14} \\
-\tau_{14} + \tau^{\#}_{12} & -\tau_{24} + \tau^{\#}_{22} & -\tau_{34} + \tau^{\#}_{32} & -\tau_{44} + \tau^{\#}_{42} & -\tau_{54} + \tau^{\#}_{52} & -\beta^{\#}_{12} - \beta_{34} & -\beta^{\#}_{22} - \beta_{44} & \beta_{14} - \beta^{\#}_{23} & \beta_{24} - \beta^{\#}_{24}
\end{array}\right] \tag{4.13}$$

On setting soul to zero we find

$$X = \left[\begin{array}{c|cc} A & 0 & 0 \\ \hline 0 & B & C \\ 0 & -C & -B^t \end{array}\right] \tag{4.14}$$

where A is real and antisymmetric while $B^\dagger = -B$ and C is complex symmetric, the Lie algebra of $\mathfrak{so}(5) \oplus \mathfrak{usp}(4)$.

The even grade subspace spanned by $\{|AB\rangle, |\bullet\bullet\rangle\}$ is acted on transitively by $\mathrm{SO}(5) \subset \mathrm{UOSp}(5|4)$ as a real 5-dimensional representation. Now recall that every physical 2-qubit state in $\mathbb{CP}^3$ is equivalent under local unitaries $\mathrm{SU}_A(2) \times \mathrm{SU}_B(2)$ to a real state:

$$|\psi\rangle \rightarrow \cos\phi|00\rangle + \sin\phi|11\rangle. \tag{4.15}$$

Hence, the bosonic SO(5) ≅ USp(4) ⊂ SU(4) subgroup of UOSp(5|4) acts transitively on the 2-qubit subspace, as required, even though it does not contain the full global unitary group SU(4). Note however, it follows from (4.10) that this SO(5) ≅ USp(4) ⊂ UOSp(5|4) cannot be identified with the USp(4) ⊂ SU(4) of quantum control under which the 2-qubit state transforms as the **4**.

4.2 Generating entangled states

To identify global UOSp(5|4) transformations not contained in the local $\mathrm{UOSp}_A(2|1) \times \mathrm{UOSp}_B(2|1)$ subgroup it is convenient to work in the 2-superqubit tensor product basis. Let

$$\tilde{G} = g \otimes g, \tag{4.16}$$

where

$$g = \left[\begin{array}{c|c} \varepsilon & 0 \\ \hline 0 & 1 \end{array}\right] = \left[\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right] \tag{4.17}$$

is the $\mathfrak{osp}(2|1)$ invariant tensor for a single superqubit. Permuting the 2-superqubit basis states into the canonical $(5|4)$ basis using a similarity transformation

$$S:\begin{array}{c}|00\rangle\\|01\rangle\\|0\bullet\rangle\\|10\rangle\\|11\rangle\\|1\bullet\rangle\\|\bullet 0\rangle\\|\bullet 1\rangle\\|\bullet\bullet\rangle\end{array}\mapsto\begin{array}{c}|00\rangle\\|01\rangle\\|10\rangle\\|11\rangle\\|\bullet\bullet\rangle\\|0\bullet\rangle\\|\bullet 0\rangle\\|1\bullet\rangle\\|\bullet 1\rangle\end{array} \tag{4.18}$$

gives

$$G = S\tilde{G}S^T = \eta \oplus \Omega, \quad \text{where} \quad \eta = \begin{bmatrix} 0 & \varepsilon & 0 \\ -\varepsilon & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{4.19}$$

Applying (3.36) we obtain a super-anti-Hermitian $X \in \mathfrak{uosp}(5|4; \mathfrak{G})$ preserving (4.19):

$$\left[\begin{array}{ccccc|cccc}
\alpha^\#_{14}-\alpha_{14} & \alpha^\#_{24}+\alpha_{13} & \alpha^\#_{34}+\alpha_{12} & 0 & -\alpha^\#_{45}-\alpha_{15} & \tau^\#_{41}-\tau_{13} & \tau^\#_{42}-\tau_{14} & \tau^\#_{43}+\tau_{11} & \tau^\#_{44}+\tau_{12} \\
-\alpha^\#_{13}-\alpha_{24} & \alpha_{23}-\alpha^\#_{23} & 0 & \alpha^\#_{34}+\alpha_{12} & \alpha^\#_{35}-\alpha_{25} & -\tau^\#_{31}-\tau_{23} & -\tau^\#_{32}-\tau_{24} & \tau_{21}-\tau^\#_{33} & \tau_{22}-\tau^\#_{34} \\
-\alpha^\#_{12}-\alpha_{34} & 0 & \alpha^\#_{23}-\alpha_{23} & \alpha^\#_{24}+\alpha_{13} & \alpha^\#_{25}-\alpha_{35} & -\tau^\#_{21}-\tau_{33} & -\tau^\#_{22}-\tau_{34} & \tau_{31}-\tau^\#_{23} & \tau_{32}-\tau^\#_{24} \\
0 & -\alpha^\#_{12}-\alpha_{34} & -\alpha^\#_{13}-\alpha_{24} & \alpha_{14}-\alpha^\#_{14} & -\alpha^\#_{15}-\alpha_{45} & \tau^\#_{11}-\tau_{43} & \tau^\#_{12}-\tau_{44} & \tau^\#_{13}+\tau_{41} & \tau^\#_{14}+\tau_{42} \\
\alpha^\#_{15}+\alpha_{45} & \alpha^\#_{25}-\alpha_{35} & \alpha^\#_{35}-\alpha_{25} & \alpha^\#_{45}+\alpha_{15} & 0 & \tau^\#_{51}-\tau_{53} & \tau^\#_{52}-\tau_{54} & \tau^\#_{53}+\tau_{51} & \tau^\#_{54}+\tau_{52} \\
\hline
-\tau^\#_{13}-\tau_{41} & \tau_{31}-\tau^\#_{23} & \tau_{21}-\tau^\#_{33} & -\tau^\#_{43}-\tau_{11} & -\tau^\#_{53}-\tau_{51} & \beta^\#_{13}-\beta_{13} & \beta^\#_{23}-\beta_{14} & \beta^\#_{33}+\beta_{11} & \beta 34^\#+\beta_{12} \\
-\tau^\#_{14}-\tau_{42} & \tau_{32}-\tau^\#_{24} & \tau_{22}-\tau^\#_{34} & -\tau^\#_{44}-\tau_{12} & -\tau^\#_{54}-\tau_{52} & \beta^\#_{14}-\beta_{23} & \beta^\#_{24}-\beta_{24} & \beta^\#_{34}+\beta_{12} & \beta^\#_{44}+\beta_{22} \\
\tau^\#_{11}-\tau_{43} & \tau^\#_{21}+\tau_{33} & \tau^\#_{31}+\tau_{23} & \tau^\#_{41}-\tau_{13} & \tau^\#_{51}-\tau_{53} & -\beta^\#_{11}-\beta_{33} & -\beta^\#_{12}-\beta_{34} & \beta_{13}-\beta^\#_{13} & \beta_{23}-\beta^\#_{14} \\
\tau^\#_{12}-\tau_{44} & \tau^\#_{22}+\tau_{34} & \tau^\#_{32}+\tau_{24} & \tau^\#_{42}-\tau_{14} & \tau^\#_{52}-\tau_{54} & -\beta^\#_{12}-\beta_{34} & -\beta^\#_{22}-\beta_{44} & \beta_{14}-\beta^\#_{23} & \beta_{24}-\beta^\#_{24}
\end{array}\right] \tag{4.20}$$

Using a straightforward reparametrisation of (4.20) the $\mathfrak{uosp}_A(2|1) \oplus \mathfrak{uosp}_B(2|1)$ subalgebra can be written as:

$$\left[\begin{array}{ccccc|cccc}
i\gamma+i\delta & \delta_++i\delta_- & \gamma_++i\gamma_- & 0 & 0 & -\sigma^\# & -\rho^\# & 0 & 0 \\
-\delta_++i\delta_- & i\gamma-i\delta & 0 & \gamma_++i\gamma_- & 0 & -\sigma & 0 & 0 & -\rho^\# \\
-\gamma_++i\gamma_- & 0 & -i\gamma+i\delta & \delta_++i\delta_- & 0 & 0 & -\rho & -\sigma^\# & 0 \\
0 & -\gamma_++i\gamma_- & -\delta_++i\delta_- & -i\gamma-i\delta & 0 & 0 & 0 & -\sigma & -\rho \\
0 & 0 & 0 & 0 & 0 & -\rho & \sigma & \rho^\# & -\sigma^\# \\
\hline
\sigma & -\sigma^\# & 0 & 0 & -\rho^\# & i\gamma & 0 & \gamma_++i\gamma_- & 0 \\
\rho & 0 & -\rho^\# & 0 & \sigma^\# & 0 & i\delta & 0 & \delta_++i\delta_- \\
0 & 0 & \sigma & -\sigma^\# & -\rho & -\gamma_++i\gamma_- & 0 & -i\gamma & 0 \\
0 & \rho & 0 & -\rho^\# & \sigma & 0 & -\delta_++i\delta_- & 0 & -i\delta
\end{array}\right] \tag{4.21}$$

where $\gamma^\#_{(\pm)} = \gamma_{(\pm)}, \delta^\#_{(\pm)} = \delta_{(\pm)}$, which follows from the graded tensor product (3.16),

$$S(x_A \otimes \mathbb{1} + \mathbb{1} \otimes x_B)S^T, \tag{4.22}$$

where

$$\begin{aligned}
x_A &= \begin{bmatrix} i\gamma & \gamma_+ + i\gamma_- & -\rho^\# \\ -\gamma_+ + i\gamma_- & -i\gamma & -\rho \\ \rho & -\rho^\# & 0 \end{bmatrix} \in \mathfrak{uosp}_A(2|1), \\
x_B &= \begin{bmatrix} i\delta & \delta_+ + i\delta_- & -\sigma^\# \\ -\delta_+ + i\delta_- & -i\delta & -\sigma \\ \sigma & -\sigma^\# & 0 \end{bmatrix} \in \mathfrak{uosp}_B(2|1).
\end{aligned} \tag{4.23}$$

A particularly simple example of a super entangled state is given by exponentiating (4.20) with only $\tau_{12} \in \mathbb{C}_a$ non-zero:

$$\exp(X_{\tau_{12}}) = \begin{bmatrix} 1+\frac{1}{2}\tau_{12}\tau_{12}^{\#} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \tau_{12} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1+\frac{1}{2}\tau_{12}\tau_{12}^{\#} & 0 & 0 & \tau_{12}^{\#} & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\tau_{12} & 0 & 0 & 1-\frac{1}{2}\tau_{12}\tau_{12}^{\#} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \tau_{12}^{\#} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1-\frac{1}{2}\tau_{12}\tau_{12}^{\#} \end{bmatrix} \tag{4.24}$$

Its action on the separable purely "bosonic" state $|00\rangle$ gives

$$\exp(X_{\tau_{12}})|00\rangle = \left(1+\frac{1}{2}\tau_{12}\tau_{12}^{\#}\right)|00\rangle - \tau_{12}^{\#}|\bullet 1\rangle, \tag{4.25}$$

where the sign on the final term follows from (3.15), which is a normalised entangled superposition of even and odd basis vectors.

The entangled state used to test Tsirelson's bound in [2],

$$|\psi\rangle = \left(1+\frac{1}{2}\tau^2+\frac{1}{2}\lambda^2+\frac{3}{4}\tau^2\lambda^2\right)[\alpha|00\rangle+\beta|11\rangle-\tau|\bullet 1\rangle-\lambda|1\bullet\rangle] \tag{4.26}$$

where $\tau^2 = \tau\tau^{\#}$, $\lambda^2 = \lambda\lambda^{\#}$ and $\alpha\alpha^{\#}+\beta\beta^{\#} = 1$, can be generated from $|00\rangle$ using a simple sequence of such transformations. This state demonstrates that using UOSp(5|4) we can generate entangled states with non-vanishing superdeterminant [1] starting from the separable $|00\rangle$ with vanishing superdeterminant. A simple set of elementary transformations of this type can also be used to generate the entangled state used in [3].

5 Conclusions

We have introduced the global super unitary group for n superqubits. For $n = 2$ we have seen that its bosonic subgroup is transitive on the 2-qubit subspace, despite the fact it does not contain the usual 2-qubit unitary group SU(4). This argument used the transitive property of USp(4) on its 5-dimensional irreducible representation spanned by $\{|AB\rangle, |\bullet\bullet\rangle\}$. The appearance of the $\mathbf{5}$ of USp(4), as opposed to the $\mathbf{4}$ encountered in the context of quantum control, is necessary since only the $\mathbf{5}$ correctly decomposes into the $(\mathbf{2},\mathbf{2})\oplus(\mathbf{1},\mathbf{1})$ under the 2-qubit local unitary group $\mathrm{SU}_A(2)\times\mathrm{SU}_B(2)$, allowing for the consistent truncation from superqubits to qubits. Using this explicit example we have seen that, starting from a separable state, UOSp(5|4) can generate states with a non-vanishing entanglement measure, the superdeterminant [1]. We will return to entanglement measures and classification elsewhere.

Let us conclude, keeping the 2-superqubit example in mind, with some comments on three or more superqubits. For three qubits we have global super unitary group UOSp(14|13). The superqubits transform as a $(\mathbf{14},\mathbf{1})\oplus(\mathbf{1},\mathbf{13})$ under the bosonic subgroup USp(14)×SO(13). As for two superqubits, the 3-qubit global unitary group SU(8) is not contained in USp(14)×SO(13), but the proper subgroup USp(8) ⊂ SU(8) is. The even states branch according as

$$\begin{array}{ccccc} \mathrm{USp}(14) & \supset & \mathrm{USp}(8) & \times & \mathrm{USp}(6) \\ \mathbf{14} & \rightarrow & (\mathbf{8},\mathbf{1}) & \oplus & (\mathbf{1},\mathbf{6}) \end{array} \tag{5.1}$$

where the $\mathbf{8}$ is spanned by $|ABC\rangle$ and the $\mathbf{6}$ by $|A\bullet\bullet\rangle, |\bullet B\bullet\rangle, |\bullet\bullet C\rangle$. Unlike two superqubits the USp(8) here is equivalent to the USp(8) of quantum control, since in both cases we have an irreducible $\mathbf{8}$. This is consistent with the truncation to three qubits as

$$\begin{array}{ccc} \mathrm{USp}(8) & \supset & \mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SU}(2) \\ \mathbf{8} & \to & (\mathbf{2},\mathbf{2},\mathbf{2}) \end{array} \tag{5.2}$$

It then follows immediately from Table 1 that the bosonic subgroup acts transitively on the subspace of 3-qubit states using the same $\mathrm{USp}(8)\subset\mathrm{SU}(8)$ as discussed in [17].

Similarly,

$$\begin{array}{ccc} \mathrm{USp}(6) & \supset & \mathrm{SU}'(2)\times\mathrm{SU}'(2)\times\mathrm{SU}'(2) \\ \mathbf{6} & \to & (\mathbf{2},\mathbf{1},\mathbf{1})\oplus(\mathbf{1},\mathbf{2},\mathbf{1})\oplus(\mathbf{1},\mathbf{1},\mathbf{2}) \end{array} \tag{5.3}$$

so that there is a diagonal $\mathrm{SU}_A(2)\times\mathrm{SU}_B(2)\times\mathrm{SU}_C(2)\subset\mathrm{USp}(14)\times\mathrm{SO}(13)$ under which,

$$\mathbf{14}\to\underbrace{(\mathbf{2},\mathbf{2},\mathbf{2})}_{|ABC\rangle}\oplus\underbrace{(\mathbf{2},\mathbf{1},\mathbf{1})}_{|A\bullet\bullet\rangle}\oplus\underbrace{(\mathbf{1},\mathbf{2},\mathbf{1})}_{|\bullet B\bullet\rangle}\oplus\underbrace{(\mathbf{1},\mathbf{1},\mathbf{2})}_{|\bullet\bullet C\rangle} \tag{5.4}$$

and, similarly for the odd $\mathbf{13}$ of SO(13),

$$\mathbf{13}\to\underbrace{(\mathbf{2},\mathbf{2},\mathbf{1})}_{|AB\bullet\rangle}\oplus\underbrace{(\mathbf{2},\mathbf{1},\mathbf{2})}_{|A\bullet C\rangle}\oplus\underbrace{(\mathbf{1},\mathbf{2},\mathbf{2})}_{|\bullet BC\rangle}\oplus\underbrace{(\mathbf{1},\mathbf{1},\mathbf{1})}_{|\bullet\bullet\bullet\rangle} \tag{5.5}$$

making clear the structure of the three superqubit states with respect to the local unitaries inside USp(14|13).

Four superqubits is the first case for which the standard global unitary group SU(16) *is* contained in the global super unitary group USp(41|40). The 4-qubit subspace spanned by $|ABCD\rangle$ transforms as the $\mathbf{16}$ of SU(16) and so question of transitivity does not appear. The standard $\mathrm{SU}(2^n)$ subgroup is present for all $n\geq 4$.

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