Advances in Applied Mathematics 110 (2019) 299–341

Contents lists available at ScienceDirect

Advances in Applied Mathematics

www.elsevier.com/locate/yaama

A construction which relates c-freeness to infinitesimal freeness

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A R T I C L E   I N F O

Article history:
Received 31 January 2019
Received in revised form 4 June 2019
Accepted 15 June 2019
Available online 2 August 2019

MSC:
primary 46L54
secondary 05A18, 60C05

A B S T R A C T

We consider two extensions of free probability that have been studied in the research literature, and are based on the notions of *c-freeness* and respectively of *infinitesimal freeness* for noncommutative random variables. In a 2012 paper, Belinschi and Shlyakhtenko pointed out a connection between these two frameworks, at the level of their operations of 1-dimensional free additive convolution. Motivated by that, we propose a construction which produces a multi-variate version of the Belinschi-Shlyakhtenko result, together with a result concerning free products of multi-variate noncommutative distributions. Our arguments are based on the combinatorics of the specific types of cumulants used in c-free and in infinitesimal free probability. They work in a rather general setting, where the initial data consists of a vector space $\mathcal{V}$ given together with a linear map $\Delta : \mathcal{V} \to \mathcal{V} \otimes \mathcal{V}$. In this setting, all the needed brands of cumulants live in the guise of families of multilinear functionals on $\mathcal{V}$, and our main

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[1] Research supported by a Discovery Grant from NSERC, Canada.
[2] Research partially supported by NCN grant 2016/23/D/ST1/01077.

https://doi.org/10.1016/j.aam.2019.06.002

result concerns a certain transformation Δ^* on such families of multilinear functionals.

1. Introduction

In this paper we study the relation between two extensions of the framework of free probability which have been considered in the research literature, and are based on the notions of *c-free independence* and respectively of *infinitesimal free independence* for noncommutative random variables.

1.1. Description of framework

We consider the plain algebraic framework of a "noncommutative probability space", or *ncps* for short, by which we simply understand a pair $(\mathcal{A}, \varphi)$ with $\mathcal{A}$ a unital algebra over $\mathbb{C}$ and $\varphi : \mathcal{A} \to \mathbb{C}$ a linear functional having $\varphi(1_{\mathcal{A}}) = 1$. The fundamental concept of interest for us in this framework is the one of *free independence* for a family of unital subalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_k \subseteq \mathcal{A}$. A handy tool for the combinatorial study of free independence is a sequence of multilinear functionals $(\kappa_n : \mathcal{A}^n \to \mathbb{C})_{n=1}^{\infty}$, called the *free cumulant functionals* associated to $(\mathcal{A}, \varphi)$; indeed, the free independence of $\mathcal{A}_1, \ldots, \mathcal{A}_k$ can be conveniently re-phrased as a vanishing condition of the "mixed" free cumulants with entries in these algebras. For a basic exposition of how free cumulants are used in the study of free independence, see e.g. Lecture 11 of [15].

In this paper we consider two extensions of the free probability framework.

One of the extensions is to the framework of *c-free* (shorthand for *"conditionally free"*) independence, which was initiated in [6,7], and has accumulated a fairly large amount of work since then (see e.g. [5,16] and the references indicated there). Here one works with triples $(\mathcal{A}, \varphi, \chi)$ where $(\mathcal{A}, \varphi)$ is an ncps (as above) and $\chi : \mathcal{A} \to \mathbb{C}$ is an additional linear functional with $\chi(1_{\mathcal{A}}) = 1$. We will refer to such a triple $(\mathcal{A}, \varphi, \chi)$ by calling it a *C-ncps*. When dealing with a C-ncps, the fundamental concept of interest is the one of *c-free independence* (with respect to φ and χ) for a family of unital subalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_k \subseteq \mathcal{A}$; this amounts to asking that $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are freely independent in the usual sense with respect to φ, and in addition, that a multiplicativity condition on χ is fulfilled for certain special products formed with elements from $\mathcal{A}_1, \ldots, \mathcal{A}_k$. The paper [7] also introduced a recipe for how to define a family $(\kappa_n^{(c)} : \mathcal{A}^n \to \mathbb{C})_{n=1}^{\infty}$ of *c-free cumulant functionals* associated to $(\mathcal{A}, \varphi, \chi)$. In terms of cumulants, the c-free independence of a family of unital subalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_k \subseteq \mathcal{A}$ is equivalent to: usual free independence of $\mathcal{A}_1, \ldots, \mathcal{A}_k$ with respect to φ, plus a vanishing condition on the mixed $\kappa_n^{(c)}$ cumulants.

The other extension we want to consider is to the framework of *infinitesimal free independence*. This was introduced in [2], with some earlier combinatorial ideas around this topic appearing in [3]. The literature on infinitesimal free probability is not extensive, but this is nevertheless a promising direction of research, e.g. due to its connections to random matrix theory ([19], see also the discussion in [14]). In order to study infinitesimal free independence, one works with triples $(\mathcal{A}, \varphi, \varphi')$ where $(\mathcal{A}, \varphi)$ is an ncps (as above) and $\varphi' : \mathcal{A} \to \mathbb{C}$ is an additional linear functional such that $\varphi'(1_{\mathcal{A}}) = 0$. We will refer to such a triple $(\mathcal{A}, \varphi, \varphi')$ by calling it an *I-ncps*. In relation to an I-ncps, the fundamental concept of interest is the one of *infinitesimal free independence* (with respect to φ and φ') for a family of unital subalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_k \subseteq \mathcal{A}$; this amounts to asking that $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are freely independent in the usual sense with respect to φ, and in addition, that a derivation-type condition on φ' is fulfilled for certain special products formed with elements from $\mathcal{A}_1, \ldots, \mathcal{A}_k$. One has, as found in [11], a natural way of defining a family $(\kappa_n' : \mathcal{A}^n \to \mathbb{C})_{n=1}^{\infty}$ of *infinitesimal free cumulant functionals* associated to $(\mathcal{A}, \varphi, \varphi')$, obtained essentially by "taking a formal derivative with respect to φ" in the formulas describing the free cumulants of φ. In terms of cumulants, the infinitesimal free independence of a family of unital subalgebras $\mathcal{A}_1, \ldots, \mathcal{A}_k \subseteq \mathcal{A}$ is equivalent to: usual free independence of $\mathcal{A}_1, \ldots, \mathcal{A}_k$ with respect to φ, plus a vanishing condition on the mixed κ_n' cumulants.

1.2. The map Ψ from [2], and its multivariate extension

The main objective of the present paper is to look for connections between the frameworks of c-free independence and of infinitesimal free independence. Our starting point is a result of Belinschi and Shlyakhtenko, Theorem 22 in [2], which goes at the level of *distributions*. For our purposes here, distributions will be assumed to have finite moments of all orders, and will be viewed as linear functionals on the algebra of polynomials $\mathbb{C}[X]$; the C-ncps and I-ncps considered in this context are thus of the form

$$(\mathbb{C}[X], \mu, \nu) \text{ and respectively } (\mathbb{C}[X], \mu, \mu'), \tag{1.1}$$

with $\mu, \nu, \mu' : \mathbb{C}[X] \to \mathbb{C}$ linear such that $\mu(1) = \nu(1) = 1$, $\mu'(1) = 0$. For pairs (μ, ν) and (μ, μ') as in (1.1) one naturally defines operations of *free additive convolution*, denoted by $\boxplus_c$ and[3] respectively $\boxplus_B$, which reflect the operation of adding c-free (respectively infinitesimally free) elements in a general C-ncps (respectively I-ncps).

Theorem 22 of [2] gives a connection between the operations of free additive convolution $\boxplus_c$ and $\boxplus_B$. It is phrased in terms of a certain map

$$\Psi : \mathcal{M} \to \mathcal{M}'$$

(introduced in the same paper [2]), where $\mathcal{M}$ is a space of probability distributions on $\mathbb{R}$ and $\mathcal{M}'$ is a space of signed measures on $\mathbb{R}$, with $\mu'(\mathbb{R}) = 0$, $\forall\, \mu' \in \mathcal{M}'$. For a $\nu \in \mathcal{M}$

[3] The second of the two notations comes from the fact that $\boxplus_B$ is also known as "free additive convolution of type B".

with compact support, the definition of $\mu' := \Psi(\nu) \in \mathcal{M}'$ can be given in the guise of a formula describing the moments of μ', namely:

$$\int_{\mathbb{R}} t^n \, d\mu'(t) = n\beta_{n+1;\nu}, \quad n \in \mathbb{N}, \tag{1.2}$$

where $(\beta_{n;\nu})_{n=1}^{\infty}$ is the sequence of *Boolean cumulants* of ν (some siblings of free cumulants, which come from the parallel world of so-called "Boolean probability"). Referring to the map Ψ, Theorem 22 of [2] can be stated as follows: if we consider probability measures $\mu_1, \nu_1, \mu_2, \nu_2 \in \mathcal{M}$ and we put

$$(\mu_1, \nu_1) \boxplus_c (\mu_2, \nu_2) = (\mu, \nu), \tag{1.3}$$

then it follows that

$$(\mu_1, \Psi(\nu_1)) \boxplus_B (\mu_2, \Psi(\nu_2)) = (\mu, \Psi(\nu)) \in \mathcal{M} \times \mathcal{M}'. \tag{1.4}$$

Clearly, one can consider a plain algebraic version of the implication "(1.3) $\Rightarrow$ (1.4)", where $\mathcal{M}$ and $\mathcal{M}'$ are replaced with spaces of linear functionals on $\mathbb{C}[X]$. In the present paper we show that, in this algebraic version, the map Ψ and the said implication "(1.3) $\Rightarrow$ (1.4)" can be generalized to a multivariate framework where instead of $\mathbb{C}[X]$ we use the algebra $\mathbb{C}\langle X_1, \ldots, X_k\rangle$ of polynomials in non-commuting indeterminates $X_1, \ldots, X_k$. More precisely: instead of $\mathcal{M}$ and $\mathcal{M}'$ we will use the spaces of "distributions" (in plain algebraic sense) $\mathcal{D}_{\text{alg}}(k)$ and $\mathcal{D}'_{\text{alg}}(k)$ defined by

$$\begin{cases} \mathcal{D}_{\text{alg}}(k) & := & \{\mu : \mathbb{C}\langle X_1, \ldots, X_k\rangle : \mu \text{ is linear, } \mu(1) = 1\}, \\ \mathcal{D}'_{\text{alg}}(k) & := & \{\mu' : \mathbb{C}\langle X_1, \ldots, X_k\rangle : \mu' \text{ is linear, } \mu'(1) = 0\}, \end{cases} \tag{1.5}$$

and we will use a map

$$\Psi_k : \mathcal{D}_{\text{alg}}(k) \to \mathcal{D}'_{\text{alg}}(k)$$

defined as follows.

Definition 1.1. Let k be a positive integer and let ν be in $\mathcal{D}_{\text{alg}}(k)$. Then $\Psi_k(\nu)$ is the linear functional $\mu' \in \mathcal{D}'_{\text{alg}}(k)$ determined by the requirement that

$$\mu'(X_{i_1} \cdots X_{i_n}) = \sum_{m=1}^{n} \beta_{n+1;\nu}(X_{i_m}, \ldots, X_{i_n}, X_{i_1}, \ldots, X_{i_m}), \tag{1.6}$$

holding for every $n \in \mathbb{N}$ and $i_1, \ldots, i_n \in \{1, \ldots, k\}$, and where $\beta_{n+1;\nu}$ denotes the $(n+1)$-th Boolean cumulant functional of ν (a multilinear functional on $\mathbb{C}\langle X_1, \ldots, X_k\rangle^{n+1}$ – the precise definition of Boolean cumulants is reviewed in Section 3 below).

It is immediate that in the case $k = 1$, Equation (1.6) boils down to (1.2).

For general $k \in \mathbb{N}$, the link between the k-variate versions of the operations $\boxplus_c$ and $\boxplus_B$ is stated in the same way as in the univariate case, with the detail that the first functional "μ" is assumed to be tracial (that is, it has the property that $\mu(PQ) = \mu(QP)$ for all $P, Q \in \mathbb{C}\langle X_1, \ldots, X_k\rangle$).

Theorem 1.2. *Let k be a positive integer and let $\mu_1, \nu_1, \mu_2, \nu_2 \in \mathcal{D}_{\text{alg}}(k)$, where μ_1 and μ_2 are tracial. If we denote*

$$(\mu_1, \nu_1) \boxplus_c (\mu_2, \nu_2) = (\mu, \nu) \in \mathcal{D}_{\text{alg}}(k) \times \mathcal{D}_{\text{alg}}(k),$$

then it follows that

$$(\mu_1, \Psi_k(\nu_1)) \boxplus_B (\mu_2, \Psi_k(\nu_2)) = (\mu, \Psi_k(\nu)) \in \mathcal{D}_{\text{alg}}(k) \times \mathcal{D}'_{\text{alg}}(k).$$

Closely related to Theorem 1.2, one has a result about the actual concepts of independence that are under discussion. At the level of multi-variate distributions, the statement of this result concerns the suitable versions of *free products* of algebraic distributions – more precisely, one calls on the free product operations "$\star_c$" and "$\star_B$" which correspond to the operations of concatenating c-free tuples (and respectively infinitesimally free tuples) in a general C-ncps (respectively I-ncps). The result about how the maps Ψ_k link these free product operations is stated as follows.

Theorem 1.3. *Let k, ℓ be positive integers and let $\mu_1, \nu_1 \in \mathcal{D}_{\text{alg}}(k)$, $\mu_2, \nu_2 \in \mathcal{D}_{\text{alg}}(\ell)$, where μ_1, μ_2 are tracial. Consider the free product*

$$(\mu_1, \nu_1) \star_c (\mu_2, \nu_2) = (\widetilde{\mu}, \widetilde{\nu}) \in \mathcal{D}_{\text{alg}}(k + \ell) \times \mathcal{D}_{\text{alg}}(k + \ell).$$

Then one has

$$(\mu_1, \Psi_k(\nu_1)) \star_B (\mu_2, \Psi_\ell(\nu_2)) = (\widetilde{\mu}, \Psi_{k+\ell}(\widetilde{\nu})) \in \mathcal{D}_{\text{alg}}(k + \ell) \times \mathcal{D}'_{\text{alg}}(k + \ell).$$

Theorems 1.2 and 1.3 are, in turn, consequences of the next result about how c-free cumulants and infinitesimal free cumulants get to be related in connection to Ψ_k.

Theorem 1.4. *Let k be a positive integer, let μ, ν be in $\mathcal{D}_{\text{alg}}(k)$ such that μ is tracial, and let $\mu' := \Psi_k(\nu) \in \mathcal{D}'_{\text{alg}}(k)$. Then the infinitesimal free cumulants $(\kappa'_n)_{n=1}^{\infty}$ of (μ, μ') are related to the c-free cumulants $(\kappa_n^{(c)})_{n=1}^{\infty}$ of (μ, ν) by the formula*

$$\kappa'_n(X_{i_1}, \ldots, X_{i_n}) = \sum_{m=1}^{n} \kappa_{n+1}^{(c)}(X_{i_m}, \ldots, X_{i_n}, X_{i_1}, \ldots, X_{i_m}),$$

holding for every $n \in \mathbb{N}$ and $i_1, \ldots, i_n \in \{1, \ldots, k\}$.

1.3. A further generalization of the map Ψ_k and of Theorem 1.4

A careful look at the proof of Theorem 1.4 reveals that the arguments needed there can be presented (and become in fact more transparent) in a framework where one simply focuses on relations between families of multilinear functionals on a vector space $\mathcal{V}$, without assuming that $\mathcal{V}$ has a multiplicative structure. In the present subsection we explain how this goes. It will come in handy to use the following notation.

Notation 1.5. Let $\mathcal{V}$ be a vector space over $\mathbb{C}$. We denote

$$\mathfrak{M}_{\mathcal{V}} := \{\underline{\varphi} : \underline{\varphi} = (\varphi_n)_{n=1}^{\infty}, \text{ with } \varphi_n : \mathcal{V}^n \to \mathbb{C} \text{ multilinear for every } n \in \mathbb{N}\}. \tag{1.7}$$

A family $\underline{\varphi} = (\varphi_n)_{n=1}^{\infty} \in \mathfrak{M}_{\mathcal{V}}$ is said to be *tracial* when it has the property that

$$\varphi_n(x_2, \ldots, x_n, x_1) = \varphi_n(x_1, \ldots, x_n), \quad \forall\, n \geq 2 \text{ and } x_1, \ldots, x_n \in \mathcal{V}. \tag{1.8}$$

Throughout a substantial part of this paper we will work with

$$\begin{cases} \text{couples } (\mathcal{V}, \underline{\varphi}) \text{ where } \underline{\varphi} \in \mathfrak{M}_{\mathcal{V}}, \text{ and} \\ \text{triples } (\mathcal{V}, \underline{\varphi}, \underline{\psi}) \text{ where } \underline{\varphi}, \underline{\psi} \in \mathfrak{M}_{\mathcal{V}}. \end{cases} \tag{1.9}$$

The motivating example for these structures is the one where $\mathcal{V}$ is a *unital algebra* and where $\underline{\varphi} = (\varphi_n)_{n=1}^{\infty}$ is completely determined by the linear functional $\varphi_1 : \mathcal{V} \to \mathbb{C}$ via the formula

$$\varphi_n(x_1, \ldots, x_n) = \varphi_1(x_1 \cdots x_n), \quad \forall\, n \geq 1 \text{ and } x_1, \ldots, x_n \in \mathcal{V}$$

(same for $\underline{\psi} = (\psi_n)_{n=1}^{\infty}$ being completely determined by $\psi_1 : \mathcal{V} \to \mathbb{C}$ in the case of a triple $(\mathcal{V}, \underline{\varphi}, \underline{\psi})$). If in the motivating example we require that $\varphi_1(1_{\mathcal{V}}) = 1$, then looking at the couple $(\mathcal{V}, \underline{\varphi})$ is pretty much the same as looking at the ncps $(\mathcal{V}, \varphi_1)$. Likewise, considering a triple $(\mathcal{V}, \underline{\varphi}, \underline{\psi})$ in the motivating example boils down to looking at $(\mathcal{V}, \varphi_1, \psi_1)$ – the latter can be either a C-ncps or an I-ncps, upon making the requirements that $\varphi_1(1_{\mathcal{V}}) = \psi_1(1_{\mathcal{V}}) = 1$, and respectively that $\varphi_1(1_{\mathcal{V}}) = 1$, $\psi_1(1_{\mathcal{V}}) = 0$.

There were four brands of cumulants appearing in the discussion of Sections 1.1 and 1.2:

- free cumulants $(\kappa_n)_{n=1}^{\infty}$ associated to a ncps $(\mathcal{A}, \varphi)$;
- Boolean cumulants $(\beta_n)_{n=1}^{\infty}$ associated to a ncps $(\mathcal{A}, \varphi)$;
- c-free cumulants $(\kappa_n^{(c)})_{n=1}^{\infty}$ associated to a C-ncps $(\mathcal{A}, \varphi, \chi)$;
- infinitesimal free cumulants $(\kappa'_n)_{n=1}^{\infty}$ associated to an I-ncps $(\mathcal{A}, \varphi, \varphi')$.

The definitions of all these four brands of cumulants go through, without any change, to the situation where instead of an ncps $(\mathcal{A}, \varphi)$ we consider a couple $(\mathcal{V}, \underline{\varphi})$ as on the first line of (1.9), and where instead of a C-ncps $(\mathcal{A}, \varphi, \chi)$ or I-ncps $(\mathcal{A}, \varphi, \varphi')$ we consider a triple $(\mathcal{V}, \underline{\varphi}, \underline{\psi})$ as on the second line of (1.9). The precise formulas for all these cumulants are reviewed in Section 3 below. A pleasing feature arising in this more general framework is that the resulting families of cumulants belong to the same space $\mathfrak{M}_{\mathcal{V}}$ that $\underline{\varphi}$ and $\underline{\psi}$ were picked from.

We now proceed to explain how the construction of the map Ψ of Belinschi-Shlyakhtenko extends to the framework of (1.9), and to present the generalization of Theorem 1.4 to this situation. We will take as input (besides the vector space $\mathcal{V}$) a linear map $\Delta : \mathcal{V} \to \mathcal{V} \otimes \mathcal{V}$, and we will use a transformation Δ^* of the space $\mathfrak{M}_{\mathcal{V}}$ which is constructed from Δ in the way described as follows.

Notation 1.6. *(The transformation Δ^*.)*

Let $\mathcal{V}$ be a vector space over $\mathbb{C}$ and let $\Delta : \mathcal{V} \to \mathcal{V} \otimes \mathcal{V}$ be a linear map.

(1) For every $n \in \mathbb{N}$ and $m \in \{1, \dots, n\}$ we let $\Delta_n^{(m)} : \mathcal{V}^{\otimes n} \to \mathcal{V}^{\otimes (n+1)}$ be the linear map determined by the requirement that

$$\Delta_n^{(m)}(x_1 \otimes \cdots \otimes x_n) = x_1 \otimes \cdots \otimes x_{m-1} \otimes \big(\Delta x_m\big) \otimes x_{m+1} \otimes \cdots \otimes x_n,$$

for all $x_1, \dots, x_n \in \mathcal{V}$. Note that in particular one has $\Delta = \Delta_1^{(1)}$.

(2) For every $n \in \mathbb{N}$ we let $\Gamma_n : \mathcal{V}^{\otimes n} \to \mathcal{V}^{\otimes n}$ be the linear map determined by the requirement that

$$\Gamma_n(x_1 \otimes \cdots \otimes x_n) = x_2 \otimes \cdots \otimes x_n \otimes x_1, \text{ for all } x_1, \dots, x_n \in \mathcal{V}.$$

In particular Γ_1 is the identity map on $\mathcal{V}$ and Γ_2 is the so-called flip map, $\Gamma_2(x_1 \otimes x_2) = x_2 \otimes x_1$.

(3) For every $n \in \mathbb{N}$ we let $\widetilde{\Delta}_n : \mathcal{V}^{\otimes n} \to \mathcal{V}^{\otimes (n+1)}$ be the linear map defined by

$$\widetilde{\Delta}_n = \sum_{m=1}^{n} \Gamma_{n+1}^m \circ \Delta_n^{(m)}.$$

[For a concrete illustration, suppose that $x_1, \dots, x_n \in \mathcal{V}$ are such that every Δx_i is a simple tensor $x_i' \otimes x_i''$, $1 \le i \le n$. Then $\widetilde{\Delta}_n(x_1 \otimes \cdots \otimes x_n)$ comes out as

$$\sum_{m=1}^{n} x_m'' \otimes x_{m+1} \otimes \cdots \otimes x_n \otimes x_1 \otimes \cdots \otimes x_{m-1} \otimes x_m'.]$$

(4) Let $\mathfrak{M}_{\mathcal{V}}$ be the space of families of multilinear functionals from Notation 1.5. We define a transformation $\Delta^* : \mathfrak{M}_{\mathcal{V}} \to \mathfrak{M}_{\mathcal{V}}$ as follows: given $\underline{\varphi} = (\varphi_n)_{n=1}^{\infty} \in \mathfrak{M}_{\mathcal{V}}$, we put

$$\Delta^*(\underline{\varphi}) := \underline{\psi} = (\psi_n)_{n=1}^{\infty} \tag{1.10}$$

where

$$\psi_n := \varphi_{n+1} \circ \widetilde{\Delta}_n, \quad \forall n \in \mathbb{N}. \tag{1.11}$$

Equation (1.11) tacitly uses some natural identifications: first, φ_{n+1} is viewed as a linear map $\mathcal{V}^{\otimes(n+1)} \to \mathbb{C}$ (rather than a multilinear map $\mathcal{V}^{n+1} \to \mathbb{C}$); this makes $\varphi_{n+1} \circ \widetilde{\Delta}_n$ be defined as a linear map $\mathcal{V}^{\otimes n} \to \mathbb{C}$, which is then identified with a multilinear map $\mathcal{V}^n \to \mathbb{C}$, as ψ_n is required to be.

Theorem 1.7. *Let $\mathcal{V}$ be a vector space over $\mathbb{C}$ and let $\Delta : \mathcal{V} \to \mathcal{V} \otimes \mathcal{V}$ be a linear map. Let $\underline{\chi} \in \mathfrak{M}_{\mathcal{V}}$, and let*

$$\underline{\varphi}' := \Delta^*(\underline{\beta}_{\underline{\chi}}), \tag{1.12}$$

with Δ^* as above and where $\underline{\beta}_{\underline{\chi}} \in \mathfrak{M}_{\mathcal{V}}$ is the family of Boolean cumulants associated to $(\mathcal{V}, \underline{\chi})$. Then for any tracial $\underline{\varphi} \in \mathfrak{M}_{\mathcal{V}}$, the following happens: denoting the c-free cumulants of $(\mathcal{V}, \underline{\varphi}, \underline{\chi})$ by $\underline{\kappa}^{(c)}$ and denoting the infinitesimal free cumulants of $(\mathcal{V}, \underline{\varphi}, \underline{\varphi}')$ by $\underline{\kappa}'$, one has the relation

$$\underline{\kappa}' = \Delta^*(\underline{\kappa}^{(c)}). \tag{1.13}$$

Remark 1.8. (1) The generalization "$\Psi_{\scriptscriptstyle\Delta}$" of the map Ψ_k from Section 1.2 is implicitly captured by Equation (1.12) in Theorem 1.7. The explicit formula for $\Psi_{\scriptscriptstyle\Delta}$ would simply be

$$\Psi_{\scriptscriptstyle\Delta}(\underline{\chi}) = \Delta^*(\underline{\beta}_{\underline{\chi}}), \quad \underline{\chi} \in \mathfrak{M}_{\mathcal{V}}.$$

(2) In order to derive Theorem 1.4 out of Theorem 1.7, one makes $\mathcal{V} = \mathbb{C}^k$ and lets Δ be the linear map defined by the prescription that

$$\Delta e_i = e_i \otimes e_i, \quad 1 \le i \le k,$$

where $e_1, \ldots, e_k$ is a fixed basis in $\mathbb{C}^k$. This prescription leads precisely to the statement of Theorem 1.4, via the natural identification of $\mathbb{C}\langle X_1, \ldots, X_k\rangle$ as the tensor algebra of $\mathbb{C}^k$. The details of how this happens are given in Section 7 below.

1.4. Organization of the paper and some related remarks

We conclude this introduction by explaining how the paper is organized, and by giving a few highlights on the content of the various sections.

Section 2 describes the combinatorial background used throughout the paper, which revolves around the lattices $NC(n)$ of non-crossing partitions. We make a brief review of some standard facts about the partial order given on $NC(n)$ by reverse refinement, and we also discuss two other partial order relations on $NC(n)$ (denoted as "$\ll$" and

"$\sqsubseteq$") which were studied in the more recent research literature, and are relevant for the considerations of the present paper.

The rest of the paper is essentially divided into two parts. The first part is set in the more general framework outlined in Section 1.3 above, and consists of Sections 3–6:

– Section 3 reviews the types of cumulants we will work with.

– In Section 4 we derive an explicit formula for c-free cumulants, which is useful for the proof of Theorem 1.7.

– Section 5 is devoted to a parallel discussion of how certain lattices of non-crossing partitions with symmetries ($NC^{(B)}(n)$ and $NC^{(B-opp)}(n)$) can be used in order to approach the cumulant functionals relevant to infinitesimally free and respectively to c-free probability. In this section we observe a simple formula, which seems to have been overlooked up to now, and helps clarifying the connection between c-free cumulants and the lattices $NC^{(B-opp)}(n)$.

– In Section 6 we obtain the proof of Theorem 1.7.

The final part of the paper consists of Sections 7 and 8, which go in the framework of multi-variable distributions considered in Section 1.2 above. In Section 7 we connect to the setting of Section 6 and we observe how Theorem 1.4 can be derived from Theorem 1.7. Then in Section 8 we show how, on the other hand, Theorem 1.4 implies Theorems 1.2 and 1.3.

2. Combinatorial background

2.1. Review of some basic $NC(n)$ combinatorics

The workhorse for the combinatorial study of free independence is the family of *lattices of non-crossing partitions* $NC(n)$. We review here some basic terminology related to this, which will be used throughout the paper. For a more detailed introduction to the $NC(n)$'s, one can for instance consult Lectures 9 and 10 of [15].

Definition 2.1. (1) Let n be a positive integer and let $\pi = \{V_1, \ldots, V_k\}$ be a partition of $\{1, \ldots, n\}$; that is, $V_1, \ldots, V_k$ are pairwise disjoint non-void sets (called the *blocks* of π) with $V_1 \cup \cdots \cup V_k = \{1, \ldots, n\}$. The number k of blocks of π will be denoted as $|\pi|$, and we will occasionally use the notation "$V \in \pi$" to mean that V is one of $V_1, \ldots, V_k$.

We say that π is *non-crossing* to mean that for every $1 \leq i_1 < i_2 < i_3 < i_4 \leq n$ such that i_1 is in the same block with i_3 and i_2 is in the same block with i_4, it necessarily follows that all of $i_1, \ldots, i_4$ are in the same block of π.

Non-crossing partitions can be naturally depicted with the numbers $1, \ldots, n$ drawn either along a line or around a circle, as illustrated in Fig. 1 below.

(2) For every $n \in \mathbb{N}$, we denote by $NC(n)$ the set of all non-crossing partitions of $\{1, \ldots, n\}$. This is one of the many combinatorial structures counted by Catalan numbers:

$$|NC(n)| = \mathrm{Cat}_n := \frac{(2n)!}{n!(n+1)!}, \quad \forall\, n \in \mathbb{N}. \tag{2.1}$$

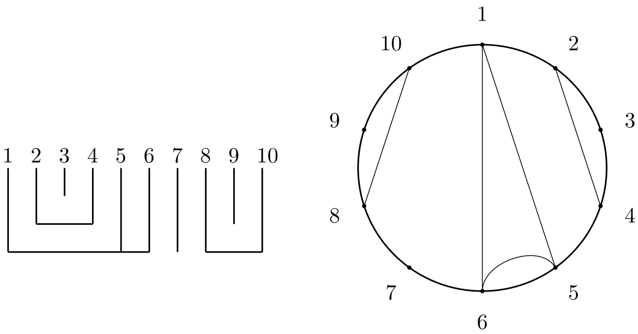


Fig. 1. The partition $\pi = \{\{1,5,6\}, \{2,4\}, \{3\}, \{7\}, \{8,10\}\,\{9\}\} \in NC(10)$, in linear representation (left) and in circular representation (right).

Definition 2.2. Let n be a positive integer. On $NC(n)$ we consider the partial order by *reverse refinement*, where for $\pi, \rho \in NC(n)$ we put

$$(\pi \leq \rho) \overset{def}{\Longleftrightarrow} \Big(\text{every block of } \rho \text{ is a union of blocks of } \pi\Big). \tag{2.2}$$

The partially ordered set $(NC(n), \leq)$ turns out to be a lattice. That is, every $\pi_1, \pi_2 \in NC(n)$ have a least common upper bound, denoted as $\pi_1 \vee \pi_2$, and have a greatest common lower bound, denoted as $\pi_1 \wedge \pi_2$.

We will use the notation 0_n for the partition of $\{1, \ldots, n\}$ into n singleton blocks and the notation 1_n for the partition of $\{1, \ldots, n\}$ into one block. It is immediate that $0_n, 1_n \in NC(n)$ and that $0_n \leq \pi \leq 1_n$ for all $\pi \in NC(n)$.

Definition and Remark 2.3. *(Kreweras complementation map.)*

One has a very useful order-reversing bijection $K_n : NC(n) \to NC(n)$, called the *Kreweras complementation map*, which provides an anti-isomorphism of the lattice $(NC(n), \leq)$. The pictorial description of K_n is given by using partitions of the set $\{1, \ldots, 2n\}$, as follows.

• For every $\pi, \sigma \in NC(n)$, let us denote by $\pi^{(\text{odd})} \sqcup \sigma^{(\text{even})}$ the partition of $\{1, \ldots, 2n\}$ which is obtained when we make π become a partition of $\{1, 3, \ldots, 2n-1\}$ and we make σ become a partition of $\{2, 4, \ldots, 2n\}$, in the natural way. That is, $\pi^{(\text{odd})} \sqcup \sigma^{(\text{even})}$ consists of blocks of the form $\{2v - 1 : v \in V\}$ where $V \in \pi$, and of blocks of the form $\{2w : w \in W\}$ where $W \in \sigma$.

• For $\pi, \sigma \in NC(n)$ it is not generally true that $\pi^{(\text{odd})} \sqcup \sigma^{(\text{even})}$ is a non-crossing partition of $\{1, \ldots, 2n\}$. If we fix $\pi \in NC(n)$, then the set

$$\{\sigma \in NC(n) : \pi^{(\text{odd})} \sqcup \sigma^{(\text{even})} \in NC(2n)\}$$

turns out to have a largest element $\sigma_{\max}$ with respect to reverse refinement order. The Kreweras complement $K_n(\pi)$ is, by definition, this special partition $\sigma_{\max}$.

[As a concrete example: for the partition π depicted in Fig. 1, one finds that $K_{10}(\pi)$ is the partition $\{ \{1,4\}, \{2,3\}, \{5\}, \{6,7,10\}, \{8,9\}, \} \in NC(10)$.

It is easily verified (see e.g. pages 146-148 in Lecture 9 of [15]) that the map $\pi \mapsto K_n(\pi)$ described above gives indeed an anti-isomorphism from $(NC(n), \leq)$ to itself.

Definition and Remark 2.4. *(Möbius function.)*

For every $n \in \mathbb{N}$, we will use the notation $\text{Möb}_n : \{(\pi, \rho) \in NC(n)^2 : \pi \leq \rho\} \to \mathbb{Z}$ for the Möbius function of the lattice $(NC(n), \leq)$, defined via the general Möbius function machinery used for any finite partially ordered set (see e.g. Chapter 3 of the monograph [20]). One can explicitly write $\text{Möb}_n(\pi, \rho)$ as a product of signed Catalan numbers (see e.g. [15], pp. 162-167 in Lecture 10); it is useful for what follows to record that one has in particular the formula

$$\text{Möb}_n(\pi, 1_n) = \text{Möb}_n(0_n, K_n(\pi)) = (-1)^{|\pi|-1} \prod_{V \in K_n(\pi)} \text{Cat}_{|V|-1}, \tag{2.3}$$

holding for every $\pi \in NC(n)$ (where the Catalan numbers $(\text{Cat}_m)_{m=1}^{\infty}$ are as reviewed in Definition 2.1, and we also make the convention to put $\text{Cat}_0 := 1$).

For the subsequent considerations related to the framework of c-free independence, it is also useful to record the following facts about how blocks of non-crossing partitions are nested inside each other.

Definition and Remark 2.5. Let n be a positive integer and let π be in $NC(n)$.

(1) Let V, W be two blocks of π. We say that V is *nested* inside W to mean that one has

$$\min(W) < \min(V) \leq \max(V) < \max(W). \tag{2.4}$$

Due to the non-crossing property of π, it is immediate that the condition (2.4) is equivalent to the apparently weaker requirement that:

$$V \neq W, \text{ and } \exists\, v \in V \text{ such that } \min(W) < v < \max(W). \tag{2.5}$$

(2) A block V of π is said to be *inner* if there exists a block W such that V is nested inside W. In the opposite case, V is said to be an *outer* block of π.

(3) Let V be an inner block of π. It is easy to check (see e.g. Proposition 2.10 in [1]) that there exists a block W of π, uniquely determined, with the properties that:

(i) V is nested inside W, and

(ii) there is no $W' \in \pi$ such that V is nested inside W' and W' is nested inside W.

We will refer to this W as the *parent-block* of V in π.

2.2. The partial order relations $\ll$ and $\sqsubseteq$

In this paper we also make use of two other partial order relations on $NC(n)$, both of them coarser than reverse refinement, which are denoted as "$\ll$" and as "$\sqsubseteq$". The partial order $\ll$ has been used for some time in free probability (starting with [1]), in the description of relations between free and Boolean cumulants. The partial order $\sqsubseteq$ is in a certain sense dual to $\ll$; it was introduced in [12], and its role was thoroughly investigated in the recent paper [4], in a more general setting which refers to the Bruhat order on Coxeter groups.

Definition and Remark 2.6. *(The partial order "$\ll$".)*

(1) For $\pi, \rho \in NC(n)$, we will write $\pi \ll \rho$ to mean that $\pi \leq \rho$ and that, in addition, for every block W of ρ there exists a block V of π such that $\min(W), \max(W) \in V$.

(2) Since in this paper we have a lot of occurrences of the special case "$\pi \ll 1_n$", let us record the obvious fact that this simply amounts to requiring π to have a unique outer block W, with $1, n \in W$. Another immediate fact which is relevant for what follows is that the Kreweras complementation map K_n maps the set $\{\pi \in NC(n) : \pi \ll 1_n\}$ onto $\{\sigma \in NC(n) : \{n\}$ is a singleton block of $\sigma\}$.

(3) In the discussion around $\ll$, a special role is played by interval partitions. A partition $\pi \in NC(n)$ is said to be an *interval partition* when every block V of π is of the form $V = [i, j] \cap \mathbb{N}$ for some $1 \leq i \leq j \leq n$. The set of all interval partitions of $\{1, \ldots, n\}$ will be denoted as $\text{Int}(n)$. It is immediate that $\text{Int}(n)$ is precisely equal to the set of maximal elements of the poset $(NC(n), \ll)$.

Remark 2.7. It will be of relevance for what follows to have some information about the structure of lower and of upper ideals of the poset $(NC(n), \ll)$. We review this here, following [1].

• *Lower ideals.* Let ρ be a fixed partition in $NC(n)$. For every block $W \in \rho$ such that $|W| \geq 3$, let us split W into the doubleton block $\{\min(W), \max(W)\}$ and $|W| - 2$ singleton blocks; when doing this, we obtain a partition $\rho_0 \leq \rho$ in $NC(n)$, such that all the blocks of ρ_0 have either 1 or 2 elements. From Definition 2.6 it is immediate that for $\pi \in NC(n)$ we have: $\pi \ll \rho \;\Leftrightarrow\; \rho_0 \leq \pi \leq \rho$. Thus the lower ideal $\{\pi \in NC(n) \;:\; \pi \ll \rho\}$ is just the interval $[\rho_0, \rho]$ with respect to reverse refinement order, for which one has a nice structure theorem (as presented for instance in [15], pages 148-150 in Lecture 9); in particular, one can explicitly write the cardinality of this lower ideal, which is

$$|\,\{\pi \in NC(n) \;:\; \pi \ll \rho\}\,| = \prod_{W \in \rho} \text{Cat}_{|W|-1}. \tag{2.6}$$

• *Upper ideals.* For a fixed $\pi \in NC(n)$, it turns out that the upper ideal $\{\rho \in NC(n) : \pi \ll \rho\}$ can be identified to a Boolean lattice, with the rank of any ρ in this lattice being equal to $|\pi| - |\rho|$ (see Proposition 2.13 and Remark 2.14 in [1]). Clearly, the Boolean lattice $\big(\{\rho \in NC(n) : \pi \ll \rho\}\,, \ll\big)$ is trivial if and only if π is maximal with respect to

$\ll$ in $NC(n)$, that is, if and only if $\pi \in \text{Int}(n)$. We record here that, as a consequence of the above, one has

$$\sum_{\substack{\rho \in NC(n) \text{ such} \\ \text{that } \pi \ll \rho}} (-1)^{|\pi|-|\rho|} = \begin{cases} 1, & \text{if } \pi \in \text{Int}(n) \\ 0, & \text{otherwise.} \end{cases} \tag{2.7}$$

Definition and Remark 2.8. *(The partial order "$\sqsubseteq$".)*

The interval refinement order $\sqsubseteq$ was introduced by Josuat-Vergès [12]: for $\pi, \rho \in NC(n)$, one writes $\pi \sqsubseteq \rho$ when $\pi \leq \rho$ and when, in addition, the partition induced by π on every block of ρ is an interval partition.

One has a form of duality between the partial orders $\sqsubseteq$ and $\ll$, implemented by the Kreweras complementation map. This was thoroughly investigated in a recent paper by Biane and Josuat-Vergés [4], in a more general setting involving the Bruhat order on Coxeter groups. (See Section 4 of [4], particularly Propositions 4.1 and 4.9.) For the reader's convenience, we provide in Lemma 2.10 below a self-contained proof of an instance of this duality which will be needed in the Section 4 of the present paper. In the proof of Lemma 2.10 we will use some natural operations "cut/attach" on non-crossing partitions, defined as follows.

Definition 2.9. Let $\pi \in NC(n)$ and let $i \in \{1, \ldots, n\}$ belong to an inner block P of π.

(1) If i is not the maximal element of P, then we define $\text{cut}(\pi, i) \in NC(n)$ to be the partition obtained from π when we cut the block P into two interval pieces in order to get that

$$Q \in \text{cut}(\pi, i) \iff (Q \in \pi \text{ and } Q \neq P) \text{ or } (Q = \{j \in P : j \leq i\}) \text{ or } (Q = \{j \in P : j > i\}).$$

(2) We define $\text{attach}(\pi, i) \in NC(n)$ to be the partition obtained from π when we attach the block P to its parent-block R (defined in the way described in Definition 2.5). That is, we have

$$Q \in \text{attach}(\pi, i) \iff (Q \in \pi \text{ and } Q \notin \{P, R\}) \text{ or } (Q = P \cup R).$$

Lemma 2.10. *The restriction of the Kreweras complementation map K_n gives an anti-isomorphism between the posets*

$$(\{\pi \in NC(n) : \pi \ll 1_n\}, \sqsubseteq) \text{ and } (\{\sigma \in NC(n) : \{n\} \in \sigma\}, \ll). \tag{2.8}$$

Proof. Consider the Hasse diagrams of the two posets indicated in the lemma (where recall that for any finite poset $(\mathcal{P}, \prec)$, the corresponding Hasse diagram is the graph with vertex-set $\mathcal{P}$ and with a "down-edge" going from p to p' if and only if p, p' are distinct elements of $\mathcal{P}$ such that $p' \prec p$ and such that there is no element $q \in \mathcal{P} \setminus \{p, p'\}$ with

$p' \prec q \prec p$). It is easy to verify that the down-edges in these two Hasse diagrams are described as follows.

• In the Hasse diagram of $(\{\pi \in NC(n) : \pi \ll 1_n\}, \sqsubseteq)$, there is a down edge from π to π' if and only if for some i in an inner block of π and not maximal in that block we have that $\pi' = \mathrm{cut}(\pi, i)$.

• In the Hasse diagram of $(\{\sigma \in NC(n) : \{n\} \in \sigma\}, \ll)$, there is a down edge from ρ to ρ' if and only if for some j in an inner block of ρ' we have $\rho = \mathrm{attach}(\rho', j)$.

We have already noticed in Remark 2.6(2) that K_n provides a bijection between the underlying sets of the two posets indicated in (2.8). In addition to that, let us make the following elementary observation, which holds for any partition $\pi \in NC(n)$ with unique outer block: if a number $i \in \{1, \ldots, n\}$ belongs to an inner block of π and is not the maximal element of that block, then i belongs to an inner block of $K_n(\pi)$ and one has

$$K_n(\mathrm{cut}(\pi, i)) = \mathrm{attach}(K_n(\pi), i). \tag{2.9}$$

Upon combining (2.9) with the explicit descriptions recorded above for the edges of the two Hasse diagrams, one finds that K_n reverses the edges in these Hasse diagrams; this implies that K_n is indeed a poset anti-isomorphism, as required. □

3. Review of four types of cumulant functionals

Throughout this section we fix a vector space $\mathcal{V}$ over $\mathbb{C}$, and we will work with the space $\mathfrak{M}_{\mathcal{V}}$ of families of multilinear functionals introduced in Notation 1.5. The goal of the section is to review the definitions of four types of cumulant functionals which we use later in the paper, in the framework of a couple $(\mathcal{V}, \underline{\varphi})$ with $\underline{\varphi} \in \mathfrak{M}_{\mathcal{V}}$ or in the framework of a triple $(\mathcal{V}, \underline{\varphi}, \underline{\psi})$ with $\underline{\varphi}, \underline{\psi} \in \mathfrak{M}_{\mathcal{V}}$. All the formulas we will use for definitions are very familiar to the people working in the area; these definitions are usually stated in the case (mentioned as "motivating example" in Section 1.3) when $\mathcal{V}$ is a unital algebra – however, nothing changes when we move to the somewhat more general framework chosen here.

Before starting, we record a customary notation which will appear in the formulas for all four types of cumulants: given an $n \in \mathbb{N}$, a tuple $(x_1, \ldots, x_n) \in \mathcal{V}^n$, and a non-empty subset $M = \{i_1, \ldots, i_m\} \subseteq \{1, \ldots, n\}$ with $i_1 < \cdots < i_m$, we denote

$$(x_1, \ldots, x_n) \mid M := (x_{i_1}, \ldots, x_{i_m}) \in \mathcal{V}^m. \tag{3.1}$$

3.1. Review of free cumulants associated to a couple $(\mathcal{V}, \underline{\varphi})$

Consider a couple $(\mathcal{V}, \underline{\varphi})$, where $\underline{\varphi} = (\varphi_n)_{n=1}^{\infty} \in \mathfrak{M}_{\mathcal{V}}$. The *free cumulants* associated to $(\mathcal{V}, \underline{\varphi})$ are the family of multilinear functionals $\underline{\kappa} = (\kappa_n)_{n=1}^{\infty} \in \mathfrak{M}_{\mathcal{V}}$ which is uniquely determined by the requirement that

$$\varphi_n(x_1,\ldots,x_n) = \sum_{\pi\in NC(n)} \prod_{V\in\pi} \kappa_{|V|}(\,(x_1,\ldots,x_n)\mid V\,), \tag{3.2}$$

holding for all $n \in \mathbb{N}$ and $x_1,\ldots,x_n \in \mathcal{V}$. The family of Equations (3.2) can be solved in order to give explicit formulas for the κ_n's, which come out as follows:

$$\kappa_n(x_1,\ldots,x_n) = \sum_{\pi\in NC(n)} \prod_{V\in\pi} \text{Möb}_n(\pi,1_n)\,\varphi_{|V|}(\,(x_1,\ldots,x_n)\mid V\,), \tag{3.3}$$

for all $n \in \mathbb{N}$ and $x_1,\ldots,x_n \in \mathcal{V}$, where $\text{Möb}_n(\pi,1_n)$ is the value of the Möbius function of $NC(n)$ which was reviewed in Remark 2.4.

3.2. Review of Boolean cumulants associated to a couple $(\mathcal{V},\underline{\varphi})$

Consider a couple $(\mathcal{V},\underline{\varphi})$, where $\underline{\varphi} = (\varphi_n)_{n=1}^{\infty} \in \mathfrak{M}_{\mathcal{V}}$. The *Boolean cumulants* associated to $(\mathcal{V},\underline{\varphi})$ are the family of multilinear functionals $\underline{\beta} = (\beta_n)_{n=1}^{\infty} \in \mathfrak{M}_{\mathcal{V}}$ which is uniquely determined by the requirement that

$$\varphi_n(x_1,\ldots,x_n) = \sum_{\pi\in \text{Int}(n)} \prod_{V\in\pi} \beta_{|V|}(\,(x_1,\ldots,x_n)\mid V\,), \tag{3.4}$$

holding for all $n \in \mathbb{N}$ and $x_1,\ldots,x_n \in \mathcal{V}$, where $\text{Int}(n)$ is the set of interval partitions reviewed in Definition 2.6(3). The family of Equations (3.4) can be solved in order to give explicit formulas for the β_n's, which come out as follows:

$$\beta_n(x_1,\ldots,x_n) = \sum_{\pi\in \text{Int}(n)} \prod_{V\in\pi} (-1)^{|\pi|+1}\varphi_{|V|}(\,(x_1,\ldots,x_n)\mid V\,), \tag{3.5}$$

for all $n \in \mathbb{N}$ and $x_1,\ldots,x_n \in \mathcal{V}$. This is analogous to how (3.2) was solved in order to obtain (3.3), with an additional simplification due to the fact that the Möbius function of $(\text{Int}(n),\leq)$ only takes the values ± 1.

3.3. Review of infinitesimal free cumulants associated to a triple $(\mathcal{V},\underline{\varphi},\underline{\varphi}')$

Consider a triple $(\mathcal{V},\underline{\varphi},\underline{\varphi}')$, where $\underline{\varphi} = (\varphi_n)_{n=1}^{\infty}$ and $\underline{\varphi}' = (\varphi'_n)_{n=1}^{\infty}$ are in $\mathfrak{M}_{\mathcal{V}}$. The *infinitesimal free cumulants* associated to $(\mathcal{V},\underline{\varphi},\underline{\varphi}')$ are the family of multilinear functionals $\underline{\kappa}' = (\kappa'_n)_{n=1}^{\infty}$ which is uniquely determined by the requirement that

$$\varphi'_n(x_1,\ldots,x_n) = \sum_{\pi\in NC(n)} \sum_{V_o\in\pi} \kappa'_{|V_o|}\big(\,(x_1,\ldots,x_n)\mid V_o\big)\cdot \prod_{\substack{V\in\pi\\ V\neq V_o}} \kappa_{|V|;\underline{\varphi}}(\,(x_1,\ldots,x_n)\mid V), \tag{3.6}$$

holding for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in \mathcal{V}$, where $\underline{\kappa}_{\underline{\varphi}} = (\kappa_{n;\underline{\varphi}})_{n=1}^{\infty}$ are the free cumulants associated to $(\mathcal{V}, \underline{\varphi})$. The family of Equations (3.6) can be solved in order to give an explicit formula for $\kappa'_n(x_1, \ldots, x_n)$, which comes out as follows:

$$\kappa'_n(x_1, \ldots, x_n) = \tag{3.7}$$
$$\sum_{\pi \in NC(n)} \sum_{V_o \in \pi} \text{Möb}_n(\pi, 1_n) \cdot \varphi'_{|V_o|}\big((x_1, \ldots, x_n) \mid V_o \big) \cdot \prod_{\substack{V \in \pi \\ V \neq V_o}} \varphi_{|V|}\big((x_1, \ldots, x_n) \mid V \big),$$

for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in \mathcal{V}$. Here $\text{Möb}_n(\pi, 1_n)$ is the same value of the Möbius function on $NC(n)$ which appeared in Equation (3.3) above.

A way to remember the formulas (3.6) and (3.7) is by noting that they are precisely what comes out when one performs a "formal derivative with respect to φ" in the formulas (3.2) and (3.3) concerning the free cumulant functionals of $(\mathcal{V}, \underline{\varphi})$. Another way of thinking about Equations (3.6) and (3.7) goes by treating the double sums on their right-hand sides as single sums over certain sets of non-crossing partitions "of type B" (cf. [11], also [10]; a brief review of this point of view is shown in Section 5.1 below).

3.4. Review of c-free cumulants associated to a triple $(\mathcal{V}, \underline{\varphi}, \underline{\chi})$

Consider a triple $(\mathcal{V}, \underline{\varphi}, \underline{\chi})$, where $\underline{\varphi} = (\varphi_n)_{n=1}^{\infty}$ and $\underline{\chi} = (\chi_n)_{n=1}^{\infty}$ are in $\mathfrak{M}_{\mathcal{V}}$. The *c-free cumulants* associated to $(\mathcal{V}, \underline{\varphi}, \underline{\chi})$ are the family of multilinear functionals $\underline{\kappa}^{(c)} = (\kappa_n^{(c)})_{n=1}^{\infty}$ which is uniquely determined by the requirement that

$$\chi_n(x_1, \ldots, x_n) = \tag{3.8}$$
$$\sum_{\pi \in NC(n)} \prod_{\substack{V \in \pi \\ V \text{ inner}}} \kappa_{|V|;\underline{\varphi}}\big((x_1, \ldots, x_n) \mid V \big) \prod_{\substack{W \in \pi \\ W \text{ outer}}} \kappa^{(c)}_{|W|}\big((x_1, \ldots, x_n) \mid W \big),$$

holding for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in \mathcal{V}$, where $\underline{\kappa}_{\underline{\varphi}} = (\kappa_{n;\underline{\varphi}})_{n=1}^{\infty}$ are the free cumulants associated to $(\mathcal{V}, \underline{\varphi})$.

The fact that $\underline{\kappa}^{(c)}$ can indeed be defined by using Equation (3.8) is easily seen when one isolates the term $\kappa_n^{(c)}(x_1, \ldots, x_n)$ indexed by the partition $1_n \in NC(n)$ on the right-hand side of (3.8); a recursive argument then shows that all the values $\kappa_n^{(c)}(x_1, \ldots, x_n)$ are uniquely determined in terms the functionals in $\underline{\chi}$ and $\underline{\kappa}_{\underline{\varphi}}$ (where the latter ones can be calculated from knowing $\underline{\varphi}$). This way of defining c-free cumulants goes back to [7], and is the one commonly used in the research literature on c-freeness.

It is not so straightforward to find some nicely structured formulas which give the $\kappa_n^{(c)}$'s explicitly, in terms of $\underline{\varphi}$ and $\underline{\chi}$ (analogously to what we had in Equations (3.3), (3.5) and (3.7) of the preceding subsections). An interesting aspect of the c-free theory is that one can still develop an approach based on certain lattices of partitions "of type

$B - opp$", found in [8], only that the cumulant functionals resulting from that approach don't coincide with the customary $\kappa_n^{(c)}$'s from (3.8). We will elaborate on this point in Section 5 below. Before that, in Section 4 we will present a direct derivation of an explicit formula relevant for the proof of our Theorem 1.7, which expresses $\kappa_n^{(c)}$ in terms of $\underline{\varphi}$ and $\underline{\beta}_{\underline{\chi}}$ (= the family of Boolean cumulants associated to $(\mathcal{V}, \underline{\chi})$).

4. An explicit formula for c-free cumulants

Throughout this section we fix a vector space $\mathcal{V}$ and two families of functionals $\underline{\varphi} = (\varphi_n)_{n=1}^\infty$ and $\underline{\chi} = (\chi_n)_{n=1}^\infty$ picked from the space $\mathfrak{M}_{\mathcal{V}}$ of Notation 1.5. We consider the family of c-free cumulants $\underline{\kappa}^{(c)} = (\kappa_n^{(c)})_{n=1}^\infty \in \mathfrak{M}_{\mathcal{V}}$ associated to the triple $(\mathcal{V}, \underline{\varphi}, \underline{\chi})$ in the way described in Section 3.4. In the present section we put into evidence an explicit formula for $\kappa_n^{(c)}(x_1, \dots, x_n)$, stated in the next proposition.

Proposition 4.1. *Consider the notations introduced above, and let us also consider the family $\underline{\beta}_{\underline{\chi}} = (\beta_{n;\underline{\chi}})_{n=1}^\infty$ of Boolean cumulants associated to $(\mathcal{V}, \underline{\chi})$. For every $n \in \mathbb{N}$ and $x_1, \dots, x_n \in \mathcal{V}$ one has:*

$$\kappa_n^{(c)}(x_1, \dots, x_n) = \tag{4.1}$$
$$\sum_{\substack{\pi \in NC(n), \\ \pi \ll 1_n}} Möb_n(\pi, 1_n) \cdot \beta_{|V_o(\pi)|;\underline{\chi}}\big((x_1, \dots, x_n) \mid V_o(\pi)\big) \cdot \prod_{\substack{V \in \pi \\ V \neq V_o(\pi)}} \varphi_{|V|}\big((x_1, \dots, x_n) \mid V\big),$$

where (following the notations from Section 2) we write "$\pi \ll 1_n$" to mean that π has a unique outer block, and where for $\pi \ll 1_n$ we denote the unique outer block of π as $V_o(\pi)$.

Proposition 4.1 solves the implicit equations (3.8) of the preceding section in a somewhat non-canonical way, which is however what is needed for going towards the connection with infinitesimal free cumulants. (As the reader may notice, the expression on the right-hand side of (4.1) bears some resemblance with the formula for free infinitesimal cumulants reviewed in Equation (3.7), with the Boolean cumulants of $\underline{\chi}$ appearing in the place where we would want to have occurrences of $\underline{\varphi}'$.)

In order to prove Proposition 4.1, we will use several lemmas.

Lemma 4.2. *Let $\underline{\kappa}_{\underline{\varphi}} = (\kappa_{n;\underline{\varphi}})_{n=1}^\infty$ and $\underline{\beta}_{\underline{\varphi}} = (\beta_{n;\underline{\varphi}})_{n=1}^\infty$ denote the free and respectively the Boolean cumulants associated to $(\mathcal{V}, \underline{\varphi})$. Then for every $n \in \mathbb{N}$ and $x_1, \dots, x_n \in \mathcal{V}$ one has*

$$\kappa_{n;\underline{\varphi}}(x_1, \dots, x_n) = \sum_{\substack{\pi \in NC(n), \\ \pi \ll 1_n}} (-1)^{|\pi|-1} \prod_{V \in \pi} \beta_{|V|;\underline{\varphi}}\big((x_1, \dots, x_n) \mid V\big). \tag{4.2}$$

Proof. This connection between free and Boolean cumulants is well-known. The derivation of Equation (4.2) can for instance be obtained by an immediate adaptation of the argument proving Proposition 3.9 of [1]. □

In a somewhat different formulation, Equation (4.3) of the next lemma was obtained in [9, Eqn. (61) in Section 6], as an application of the shuffle algebra approach to c-free cumulants developed in that paper. We present here a direct proof of Equation (4.3), relying on some basic properties of the partial order $\ll$ on $NC(n)$.

Lemma 4.3. *Consider the framework and notations introduced above. For every $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in \mathcal{V}$, one has:*

$$\kappa_n^{(c)}(x_1, \ldots, x_n) = \sum_{\substack{\pi \in NC(n), \\ \pi \ll 1_n}} (-1)^{|\pi|-1} \cdot \beta_{|V_o(\pi)|;\underline{\chi}}(\,(x_1, \ldots, x_n) \mid V_o) \cdot \prod_{\substack{V \in \pi \\ V \neq V_o(\pi)}} \beta_{|V|;\underline{\varphi}}(\,(x_1, \ldots, x_n) \mid V). \tag{4.3}$$

Proof. For every $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in \mathcal{V}$, we denote the right-hand side of (4.3) as $\lambda_n(x_1, \ldots, x_n)$. In this way we define a family of multilinear functionals $\underline{\lambda} = (\lambda_n)_{n=1}^{\infty} \in \mathfrak{M}_{\mathcal{V}}$, and the statement of the lemma amounts to proving that $\underline{\lambda} = \underline{\kappa}^{(c)}$. In order to obtain this equality, it suffices to prove that $\underline{\lambda}$ satisfies the family of equations (3.8) which were used to define $\underline{\kappa}^{(c)}$. That is, we have to verify the family of equalities of the form

$$\chi_n(x_1, \ldots, x_n) = \sum_{\pi \in NC(n)} \prod_{\substack{V \in \pi \\ V \text{ inner}}} \kappa_{|V|;\underline{\varphi}}(\,(x_1, \ldots, x_n) \mid V) \prod_{\substack{W \in \pi \\ W \text{ outer}}} \lambda_{|W|}(\,(x_1, \ldots, x_n) \mid W). \tag{4.4}$$

For the remaining part of the proof we fix an $n \in \mathbb{N}$ and some $x_1, \ldots x_n \in \mathcal{V}$, for which we will verify that (4.4) holds.

We start from the sum on the right-hand side of (4.4). Let us fix for the moment a $\pi \in NC(n)$, and let us focus on the term indexed by π in the said sum. This term can be itself written as a sum, if we take the following steps:

$$\begin{cases} \text{– for every inner block } V \text{ of } \pi, \text{ we replace } \kappa_{|V|;\underline{\varphi}}(\,(x_1, \ldots, x_n) \mid V)\,) \text{ as a} \\ \text{sum } \sum_V \text{ indexed by } \{\rho_V \in NC(|V|) : \rho_V \text{ has unique outer block}\}, \text{ by} \\ \text{using Lemma 4.2;} \\ \text{– for every outer block } W \text{ of } \pi, \text{ we replace } \lambda_{|W|}(\,(x_1, \ldots, x_n) \mid W)\,) \text{ as} \\ \text{a sum } \sum_W \text{ indexed by } \{\rho_W \in NC(|W|) : \rho_W \text{ has unique outer block}\}, \\ \text{by using the definition of } \lambda_{|W|}; \\ \text{– we cross-multiply all the sums } \sum_V \text{ and } \sum_W \text{ found in the preceding} \\ \text{two steps.} \end{cases} \tag{4.5}$$

If one chooses a partition ρ_V for every inner block $V \in \pi$ and a partition ρ_W for every outer block $W \in \pi$ in the way indicated in (4.5), then putting all these ρ_V, ρ_W together results in a partition $\rho \in NC(n)$ such that $\rho \ll \pi$, with "$\ll$" being the partial order discussed in Remark 2.6. We leave it as an exercise to the reader to write down the general term of the summation over $\{\rho \in NC(n) : \rho \ll \pi\}$ which is produced by the third step of (4.5), and to conclude that what one gets is the following formula:

$$\prod_{\substack{V\in\pi \\ V \text{ inner}}} \kappa_{|V|;\underline{\varphi}}\big((x_1,\ldots,x_n) \mid V\big) \prod_{\substack{W\in\pi \\ W \text{ outer}}} \lambda_{|W|}\big((x_1,\ldots,x_n) \mid W\big) \tag{4.6}$$

$$= \sum_{\rho \ll \pi} (-1)^{|\rho|-|\pi|} \prod_{\substack{V\in\rho \\ V \text{ inner}}} \beta_{|V|;\underline{\varphi}}\big((x_1,\ldots,x_n) \mid V\big) \prod_{\substack{W\in\rho \\ W \text{ outer}}} \beta_{|W|;\underline{\chi}}\big((x_1,\ldots,x_n) \mid W\big).$$

We now let π run in $NC(n)$. Returning to the equality to be proved, Equation (4.4), we see that its right-hand side (obtained by summing over π in (4.6)) is equal to

$$\sum_{\pi\in NC(n)} \sum_{\rho \ll \pi} (-1)^{|\rho|-|\pi|} \prod_{\substack{V\in\rho \\ V \text{ inner}}} \beta_{|V|;\underline{\varphi}}\big((x_1,\ldots,x_n) \mid V\big) \cdot \prod_{\substack{W\in\rho \\ W \text{ outer}}} \beta_{|W|;\underline{\chi}}\big((x_1,\ldots,x_n) \mid W\big). \tag{4.7}$$

We must verify that the quantity in (4.7) is equal to $\chi_n(x_1,\ldots,x_n)$. To this end, we exchange the order of summation over π and ρ, so that (4.7) becomes:

$$\sum_{\rho\in NC(n)} \Bigg(\sum_{\substack{\pi\in NC(n), \\ \pi \gg \rho}} (-1)^{|\rho|-|\pi|} \Bigg) \prod_{\substack{V\in\rho \\ V \text{ inner}}} \beta_{|V|;\underline{\varphi}}\big((x_1,\ldots,x_n) \mid V\big) \times$$
$$\times \prod_{\substack{W\in\rho \\ W \text{ outer}}} \beta_{|W|;\underline{\chi}}\big((x_1,\ldots,x_n) \mid W\big). \tag{4.8}$$

It was however noticed in Section 2 that for a fixed $\rho \in NC(n)$ one has:

$$\sum_{\substack{\pi\in NC(n), \\ \pi \gg \rho}} (-1)^{|\rho|-|\pi|} = \begin{cases} 1 & \text{if } \rho \in \mathrm{Int}(n), \\ 0 & \text{otherwise.} \end{cases}$$

Therefore the summation in (4.8) reduces to just $\sum_{\rho\in\mathrm{Int}(n)} \prod_{W\in\rho} \beta_{|W|;\underline{\chi}}\big((x_1,\ldots,x_n) \mid W\big)$ (where we also took into account that all blocks of an interval partition are outer blocks). The latter sum is indeed equal to $\chi_n(x_1,\ldots,x_n)$, as required, by the definition of Boolean cumulants. □

Lemma 4.4. *Let $n \geq 2$ be an integer, let V_o be a subset of $\{1,\ldots,n\}$ such that $V_o \ni 1, n$, and let $x_1,\ldots,x_n$ be in $\mathcal{V}$. One has:*

$$\sum_{\substack{\pi\in NC(n)\ such\\ that\ V_o\in\pi}} (-1)^{1+|\pi|} \prod_{\substack{V\in\pi,\\ V\neq V_o}} \beta_{|V|;\underline{\varphi}}\big((x_1,\ldots,x_n)\mid V\big) \tag{4.9}$$

$$= \sum_{\substack{\rho\in NC(n)\ such\\ that\ V_o\in\rho}} M\ddot{o}b_n(\rho,1_n)\cdot \prod_{\substack{V\in\rho\\ V\neq V_o}} \varphi_{|V|}\big((x_1,\ldots,x_n)\mid V\big).$$

Proof. Fix for the moment a partition $\pi \in NC(n)$ such that $V_o \in \pi$. For every block $V \neq V_o$ of π, we use the definition of Boolean cumulants in order to express $\beta_{|V|;\underline{\varphi}}((x_1,\ldots,x_n) \mid V)$ as a sum indexed by $\mathrm{Int}(|V|)$, and then we cross-multiply the resulting sums. The reader should have no difficulty to verify that, upon doing this cross-multiplication, one arrives to the formula

$$\prod_{\substack{V\in\pi,\\ V\neq V_o}} \beta_{|V|}\big((x_1,\ldots,x_n)\mid V\big) = \sum_{\substack{\rho\sqsubseteq\pi\ such\\ that\ V_o\in\rho}} (-1)^{|\rho|-|\pi|} \prod_{\substack{V\in\rho,\\ V\neq V_o}} \varphi\big((x_1,\ldots,x_n)\mid V\big), \tag{4.10}$$

where $\sqsubseteq$ is the partial order relation on $NC(n)$ reviewed in Section 2.2.

We now let π run in the index set shown on the left-hand side of (4.9). By summing in Equation (4.10) over this range for π, we find that

$$\sum_{\substack{\pi\in NC(n)\ such\\ that\ V_o\in\pi}} (-1)^{|\pi|-1} \prod_{V\in\pi,\, V\neq V_o} \beta_{|V|}\big((a_1,\ldots,a_n)\mid V\big)$$

$$= \sum_{\substack{\pi\in NC(n)\ such\\ that\ V_o\in\pi}} \sum_{\substack{\rho\sqsubseteq\pi\ such\\ that\ V_o\in\rho}} (-1)^{|\rho|-1} \prod_{V\in\rho,\, V\neq V_o} \varphi\big((a_1,\ldots,a_n)\mid V\big).$$

By exchanging sums indexed by ρ and π, we can continue the above with

$$= \sum_{\substack{\rho\in NC(n)\ such\\ that\ V_o\in\rho}} (-1)^{|\rho|-1} |\{\pi\in NC(n) : \rho\sqsubseteq\pi\}| \cdot \prod_{\substack{V\in\rho\\ V\neq V_o}} \varphi_{|V|}\big((a_1,\ldots,a_n)\mid V\big).$$

In order to conclude the proof, we are left to verify that for every $\rho \in NC(n)$ such that $V_o \in \rho$, one has

$$(-1)^{|\rho|-1} \, |\{\pi \in NC(n) : \rho \sqsubseteq \pi\}| = \text{Möb}_n(\rho, 1_n).$$

To this end, we invoke Lemma 2.10, which gives us that

$$|\{\pi \in NC(n) : \rho \sqsubseteq \pi\}| = |\{\pi \in NC(n) : \pi \ll \rho\}| \, .$$

It was noticed in Section 2 that the latter cardinality is equal to the product of Catalan numbers $\prod_{V \in \rho} C_{|V|-1}$, and upon multiplying this with $(-1)^{|\rho|-1}$, one arrives indeed to the required value $\text{Möb}_n(\rho, 1_n)$. □

Proof of Proposition 4.1. The case when $n = 1$ is clear (both sides of Equation (4.1) are equal to $\chi_1(x_1)$). For $n \geq 2$, we start from the expression for $\kappa_n^{(c)}(x_1, \ldots, x_n)$ found in Lemma 4.3, where we write the sum on the right-hand side of Equation (4.3) as a double sum of the form

$$\sum_{\substack{V_o \subseteq \{1,\ldots,n\} \\ with\ V_o \ni 1,n}} \; \sum_{\substack{\pi \in NC(n)\ such \\ that\ V_o \in \pi}} [\text{term indexed by } V_o \text{ and } \pi]. \tag{4.11}$$

Then for every $V_o \subseteq \{1, \ldots, n\}$ such that $V_o \ni 1, n$ we replace the second sum in (4.11) by using Lemma 4.4. When doing this replacement we arrive to a double sum of the form

$$\sum_{\substack{V_o \subseteq \{1,\ldots,n\} \\ with\ V_o \ni 1,n}} \; \sum_{\substack{\rho \in NC(n)\ such \\ that\ V_o \in \rho}} [\text{term indexed by } V_o \text{ and } \rho],$$

and converting the latter double sum into one sum indexed by $\{\rho \in NC(n) : \rho \ll 1_n\}$ leads to the required formula (4.11). □

5. Non-crossing partitions with symmetries, and a variation on c-free cumulant functionals

The "typical" combinatorial approach to any brand of cumulants goes by identifying a coherent sequence of *lattices of partitions*, which are used as index sets for the various summation formulas describing the cumulants in question. This applies in particular to the free cumulants and to the Boolean cumulants reviewed in Sections 3.1 and 3.2, where the relevant lattices of partitions are $\{NC(n) : n \in \mathbb{N}\}$ and respectively $\{\text{Int}(n) : n \in \mathbb{N}\}$. The approach with partitions is good because it offers streamlined ideas on how to proceed – for instance one typically uses (following a method initiated by Rota in the

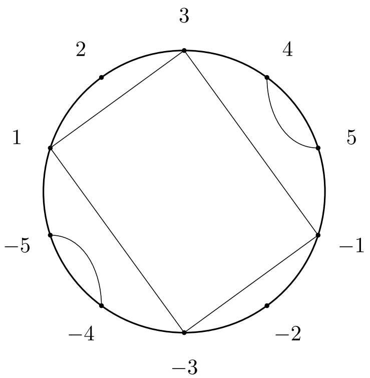


Fig. 2. The partition $\{ \{1, 3, -1, -3\}, \{2\}, \{-2\}, \{4, 5\}, \{-4, -5\} \} \in NC^{(B)}(5)$.

1960's) a notion of "multiplicative functions" on the relevant sequence of lattices, then one looks for a formula for "cumulants with products as entries", where an essential ingredient in the formula is the use of lattice operations (see e.g. Section 3.2 of [13]).

In this section we review the approach to cumulants via partitions in the two extended frameworks we are considering: c-free cumulants and infinitesimal free cumulants, and we point out a relevant fact which seems to have been overlooked in the c-free case.

5.1. Lattices of partitions for infinitesimal free cumulants

We first go over the (better understood) case of infinitesimal free cumulants. In order to approach these cumulants via the study of multiplicative functions on a sequence of lattices, one uses an "analogue of type B" for the $NC(n)$'s; this is a family of lattices denoted as $NC^{(B)}(n)$ which were introduced in [17]. In order to depict a partition $\sigma \in NC^{(B)}(n)$, one starts by marking on a circle $2n$ points, labeled as $1, \ldots, n$ and $-1, \ldots, -n$ (as exemplified, for $n = 5$, in Fig. 2 above). Then σ must achieve a non-crossing partition of these $2n$ points, with the additional symmetry requirement that if U is a block of σ then $-U := \{-i : i \in U\}$ is a block of σ as well.

Note that a block U of a $\sigma \in NC^{(B)}(n)$ either is such that $U = -U$, in which case we say that U is a *zero-block* of σ, or is such that $U \cap (-U) = \emptyset$. The non-crossing condition forces every $\sigma \in NC^{(B)}(n)$ to have at most one zero-block; moreover, there exists a natural bijection between $\{\sigma \in NC^{(B)}(n) : \sigma \text{ has a zero-block}\}$ and $\{\sigma \in NC^{(B)}(n) : \sigma \text{ has no zero-block}\}$, which is implemented by a suitable version of the Kreweras complementation map.

In the papers [3] and then [11] it was pointed out that, when used on the lattices $NC^{(B)}(n)$, the general machinery of Rota leads to the infinitesimal free cumulant functionals $\kappa'_n : \mathcal{A}^n \to \mathbb{C}$ associated to an I-ncps $(\mathcal{A}, \varphi, \varphi')$. The same considerations apply, without any changes, to the framework of $(\mathcal{V}, \underline{\varphi}, \underline{\varphi}')$ used in Section 3.3. Since the families of equations (3.6) and (3.7) shown in Section 3.3 did not, at the face of it, call on the lattices $NC^{(B)}(n)$, it is of relevance to mention here that the double sum "$\sum_{\pi \in NC(n)} \sum_{V \in \pi}$"

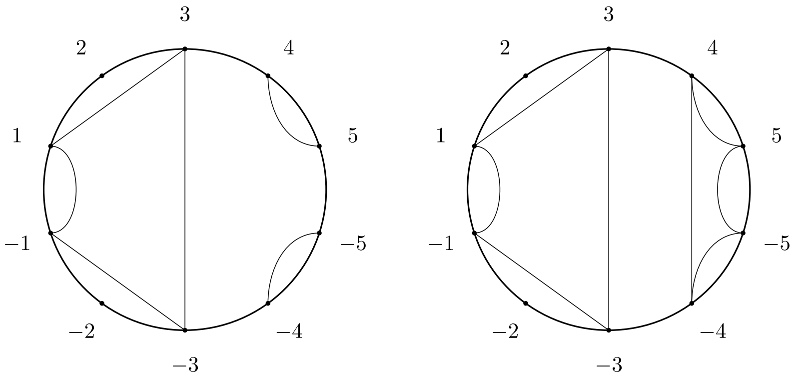


Fig. 3. The partitions $\{\{1, 3, -1, -3\}, \{2\}, \{-2\}, \{4, 5\}, \{-4, -5\}\}$ (left) and $\{\{1, 3, -1, -3\}, \{2\}, \{-2\}, \{4, 5, -4, -5\}\}$ (right) in $NC^{(B-opp)}(5)$.

appearing in these families of equations comes from processing a single sum indexed by $\{\sigma \in NC^{(B)}(n) : \sigma$ has a zero-block$\}$. For instance, Equation (3.6) corresponds to a formula which looks like this:

$$\varphi'_n(x_1, \ldots, x_n) = \sum_{\substack{\sigma \in NC^{(B)}(n) \\ with\ zero-block\ Z}} \kappa'_{|Z|/2}\big((x_1, \ldots, x_n) \mid \mathrm{Abs}(Z)\big) \times \tag{5.1}$$

$$\times \prod_{\substack{pairs\ U, -U \in \sigma \\ such\ that\ U \neq (-U)}} \kappa_{|U|;\underline{\varphi}}\big((x_1, \ldots, x_n) \mid \mathrm{Abs}(U)\big),$$

where $\mathrm{Abs} : \{1, \ldots, n\} \cup \{-1, \ldots, -n\} \to \{1, \ldots, n\}$ is the absolute value map sending $\pm i$ to i for $1 \le i \le n$. (For the details of how one passes back-and-forth between Equations (3.6) and (5.1) we refer the reader to Section 6 of [11].)

5.2. Lattices of partitions for conditionally free cumulants

Now let us look at the *c*-free framework. In order to approach cumulants via the study of multiplicative functions on a sequence of lattices, one uses here some lattices $NC^{(B-opp)}(n)$ which were identified in [8]. The drawing of a $\sigma \in NC^{(B-opp)}(n)$ starts by marking on a circle $2n$ points, labeled as $1, \ldots, n$ and $-n, \ldots, -1$ (as exemplified, for $n = 5$, in Fig. 3 above). Then σ must achieve a non-crossing partition of these $2n$ points, with the additional symmetry requirement that if U is a block of σ then $-U := \{-i : i \in U\}$ is a block of σ as well. Note that the description of how we obtain our σ is strikingly similar to the description at the beginning of Section 5.1 – the only difference (and the reason for using the name "$NC^{(B-opp)}(n)$") is that the points with negative labels $-1, \ldots, -n$ now appear *in reverse order* as we travel around the circle.

Similar to how things went for partitions in $NC^{(B)}(n)$, a block U of a partition σ in $NC^{(B-opp)}(n)$ either is such that $U = -U$, in which case we say that U is a *zero-block* of σ, or is such that $U \cap (-U) = \emptyset$. But unlike how things went for $NC^{(B)}(n)$, a partition $\sigma \in NC^{(B-opp)}(n)$ may have multiple zero-blocks (for instance the second partition shown in Fig. 3 has two such blocks).

In the paper [8] it was shown that, when used on the lattices $NC^{(B-opp)}(n)$, the general machinery of Rota leads to the identification of a family of cumulants $(\kappa_n^{(cc)} : \mathcal{A}^n \to \mathbb{C})_{n=1}^{\infty}$ associated to a C-ncps $(\mathcal{A}, \varphi, \chi)$, which can be used for describing c-freeness for subalgebras of $\mathcal{A}$. A puzzling detail appearing at this point is that the functionals $\kappa_n^{(cc)}$ are not the same as the $\kappa_n^{(c)}$'s coming from [7], which were reviewed in Section 3.4. The main goal of the present section is to resolve this puzzle by pointing out a neat direct connection between the $\kappa_n^{(c)}$'s and $\kappa_n^{(cc)}$'s, in Proposition 5.4 below. We start towards that by recording a few elementary observations about the structure of the partitions in $NC^{(B-opp)}(n)$.

Remark 5.1. Let σ be a partition in $NC^{(B-opp)}(n)$.

(1) If U is a block of σ such that $U \cap \{1, \ldots, n\} \neq \emptyset \neq U \cap \{-1, \ldots, -n\}$, then U must be a zero-block. Indeed, let $i, j \in \{1, \ldots, n\}$ be such that $i, -j \in U$. If $i = j$, then $i \in U \cap (-U)$, which forces $U = -U$. If $i \neq j$, then we look at the four points $i, j, -i, -j$ drawn around the circle and we observe that if U and $-U$ would be distinct blocks of σ, then there would be crossings between them (e.g. if $i < j$, then the four points we're looking at come in the order $i, j, -j, -i$, with $i, -j \in U$ and $j, -i \in -U$). So this case, too, leads to the conclusion that $U = -U$.

(2) The contrapositive of (1) is that every non-zero-block of σ either is contained in $\{1, \ldots, n\}$ or is contained in $\{-1, \ldots, -n\}$. Thus the non-zero blocks of σ come in pairs $U, -U$, with one of $U, -U$ contained in $\{1, \ldots, n\}$ and the other contained in $\{-1, \ldots, -n\}$.

Definition and Remark 5.2. Let $\mathcal{V}$ be a vector space over $\mathbb{C}$, and consider (analogous to the considerations of Section 3.4) a triple $(\mathcal{V}, \underline{\varphi}, \underline{\chi})$ where $\underline{\varphi} = (\varphi_n)_{n=1}^{\infty}$ and $\underline{\chi} = (\chi_n)_{n=1}^{\infty}$ are families of multilinear functionals on $\mathcal{V}$. We will use the name of *alternative c-free cumulants* associated to $(\mathcal{V}, \underline{\varphi}, \underline{\chi})$ for the family of multilinear functionals $\underline{\kappa}^{(cc)} = (\kappa_n^{(cc)})_{n=1}^{\infty}$ which is uniquely determined by the requirement that

$$\chi_n(x_1, \ldots, x_n) = \sum_{\sigma \in NC^{(B-opp)}(n)} \prod_{\substack{U \in \sigma, \\ U \subseteq \{1,\ldots,n\}}} \kappa_{|U|;\underline{\varphi}}((x_1, \ldots, x_n) \mid U) \prod_{\substack{Z \in \sigma, \\ Z = -Z}} \kappa_{|Z|/2}^{(cc)}((x_1, \ldots, x_n) \mid Z \cap \{1, \ldots, n\}), \tag{5.2}$$

holding for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in \mathcal{V}$, and where $(\kappa_{n;\underline{\varphi}})_{n=1}^{\infty}$ are the free cumulants associated to $(\mathcal{V}, \underline{\varphi})$.

The fact that $\underline{\kappa}^{(cc)}$ can indeed be defined by using Equation (5.2) is easily seen, in a similar way to how it was seen that the c-free cumulants $\underline{\kappa}^{(c)}$ associated to $(\mathcal{V}, \underline{\varphi}, \underline{\chi})$ are correctly defined by Equation (3.8) of Section 3.4. In the case at hand, one isolates on the right-hand side of (5.2) the term $\kappa_n^{(cc)}(x_1, \ldots, x_n)$ which is indexed by the partition $\sigma \in NC^{(B-opp)}$ with only one block, and then one proceeds by induction.

Before stating the proposition which relates $\underline{\kappa}^{(cc)}$ to $\underline{\kappa}^{(c)}$ we record another remark about $NC^{(B-opp)}$, concerning the natural "absolute value map" $\mathrm{Abs} : NC^{(B-opp)}(n) \to NC(n)$.

Remark and Notation 5.3. Let n be a positive integer, and for every subset $U \subseteq \{1, \ldots, n\} \cup \{-1, \ldots, -n\}$, let us agree to denote $\mathrm{Abs}(U) := \{|i| : i \in U\}$.

(1) It is immediate that if σ is a partition in $NC^{(B-opp)}(n)$ and if U_1, U_2 are two blocks of σ, then either $\mathrm{Abs}(U_1) = \mathrm{Abs}(U_2)$ or $\mathrm{Abs}(U_1) \cap \mathrm{Abs}(U_2) = \emptyset$. This implies that the set of sets

$$\mathrm{Abs}(\sigma) := \{\mathrm{Abs}(U) : U \in \sigma\}$$

is a partition of $\{1, \ldots, n\}$. It is, moreover, easily seen that $\mathrm{Abs}(\sigma)$ must belong to $NC(n)$. Indeed, for any two distinct blocks V_1, V_2 of $\mathrm{Abs}(\sigma)$ it is possible to pick two distinct blocks U_1, U_2 of σ such that $V_1 \subseteq U_1$ and $V_2 \subseteq U_2$; so a crossing between V_1 and V_2 would entail a crossing between U_1 and U_2, which is not possible.

(2) Let σ be in $NC^{(B-opp)}(n)$, and consider the partition $\pi = \mathrm{Abs}(\sigma) \in NC(n)$. We note that if Z is a zero-block of σ, then $W := \mathrm{Abs}(Z)$ has to be an outer block of π. Indeed, if W was to be nested inside some other block V of π, then upon writing $V = \mathrm{Abs}(U)$ for some $U \in \sigma$ one would immediately find crossings between Z and U, which is not possible.

(3) Parts (1) and (2) of this remark provide us with a map

$$\sigma \mapsto \Big(\mathrm{Abs}(\sigma), \, \{\mathrm{Abs}(Z) : Z \in \sigma, \; Z = -Z\} \Big) \tag{5.3}$$

going from $NC^{(B-opp)}(n)$ to the set $\{(\pi, \mathcal{S}) : \pi \in NC(n), \, \mathcal{S} \subseteq \mathrm{Out}(\pi)\}$, where we used the notation $\mathrm{Out}(\pi)$ for the set of all outer blocks of a partition $\pi \in NC(n)$. We leave it as an exercise to the reader to check that the map (5.3) is a bijection, with inverse described as follows: given $\pi \in NC(n)$ and a set $\mathcal{S}$ (possibly empty) of outer blocks of π, we put

$$\sigma := \{W \cup (-W) : W \in \mathcal{S}\} \cup \{V : V \in \pi \setminus \mathcal{S}\} \cup \{-V : V \in \pi \setminus \mathcal{S}\}$$

(where $\pi \setminus \mathcal{S}$ denotes the set of blocks of π not taken in $\mathcal{S}$).

Proposition 5.4. *Let $\mathcal{V}$ be a vector space over $\mathbb{C}$, and consider a triple $(\mathcal{V}, \underline{\varphi}, \underline{\chi})$ where $\underline{\varphi} = (\varphi_n)_{n=1}^{\infty}$ and $\underline{\chi} = (\chi_n)_{n=1}^{\infty}$ are in $\mathfrak{M}_{\mathcal{V}}$. Let $\underline{\kappa}^{(c)} = (\kappa_n^{(c)})_{n=1}^{\infty}$ be the c-free cumulants* *associated to* $(\mathcal{V}, \underline{\varphi}, \underline{\chi})$ *as in Section 3.4, and let* $\underline{\kappa}^{(cc)} = (\kappa_n^{(cc)})_{n=1}^{\infty}$ *be the alternative c-free cumulants considered in Definition 5.2. One has*

$$\kappa_n^{(cc)} = \kappa_n^{(c)} - \kappa_{n;\underline{\varphi}}, \quad n \in \mathbb{N}, \tag{5.4}$$

where $(\kappa_{n;\underline{\varphi}})_{n=1}^{\infty}$ *are the free cumulants associated to* $(\mathcal{V}, \underline{\varphi})$.

Proof. The bijection observed in Remark 5.3(3) can be used as a change of variable in the Equation (5.2) defining $\underline{\kappa}^{(cc)}$, which then takes the form:

$$\chi_n(x_1, \ldots, x_n) = \sum_{\pi \in NC(n)} \sum_{\mathcal{S} \subseteq \text{Out}(\pi)} \prod_{V \in \pi \setminus \mathcal{S}} \kappa_{|V|;\underline{\varphi}}(\,(x_1, \ldots, x_n) \mid V) \cdot \prod_{W \in \mathcal{S}} \kappa_{|W|}^{(cc)}(\,(x_1, \ldots, x_n) \mid W), \tag{5.5}$$

holding for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in \mathcal{V}$. It is immediate that if on the right-hand side of (5.5) we fix a $\pi \in NC(n)$ and we only perform the sum over $\mathcal{S} \subseteq \text{Out}(\pi)$, what results is the product

$$\prod_{\substack{V \in \pi \\ V \text{ inner}}} \kappa_{|V|;\underline{\varphi}}((x_1, \ldots, x_n) \mid V) \cdot \prod_{\substack{W \in \pi \\ W \text{ outer}}} \big(\kappa_{|W|;\underline{\varphi}}((x_1, \ldots, x_n) \mid W) + \kappa_{|W|}^{(cc)}((x_1, \ldots, x_n) \mid W)\big). \tag{5.6}$$

For every $n \in \mathbb{N}$ let us put $\lambda_n := \kappa_n^{(cc)} + \kappa_{n;\underline{\varphi}}$, and let $\underline{\lambda} := (\lambda_n)_{n=1}^{\infty} \in \mathfrak{M}_{\mathcal{V}}$. Upon replacing (5.6) inside the right-hand side of Equation (5.5), we find that

$$\chi_n(x_1, \ldots, x_n) = \sum_{\pi \in NC(n)} \prod_{\substack{V \in \pi \\ V \text{ inner}}} \kappa_{|V|;\underline{\varphi}}((x_1, \ldots, x_n) \mid V) \cdot \prod_{\substack{W \in \pi \\ W \text{ outer}}} \lambda_{|W|}(\,(x_1, \ldots, x_n) \mid W), \tag{5.7}$$

holding for every $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in \mathcal{V}$. We have thus obtained that $\underline{\lambda}$ satisfies the family of equations (3.8) which were used to define $\underline{\kappa}^{(c)}$. It follows that $\underline{\lambda} = \underline{\kappa}^{(c)}$, which concludes the proof. □

Remark 5.5. (1) Let $(\mathcal{A}, \varphi, \chi)$ be a C-ncps, and let $\mathcal{A}_1, \ldots, \mathcal{A}_k$ be unital subalgebras of $\mathcal{A}$. Proposition 5.4 explains in a rather neat way why the functionals $\kappa_n^{(cc)}$ can indeed be used in the description of c-free independence, in a similar way to how the $\kappa_n^{(c)}$ are used. Indeed, in the presence of the background condition that $\mathcal{A}_1, \ldots, \mathcal{A}_k$ are freely independent with respect to φ (which imposes the vanishing of the mixed free cumulants with respect to φ), Proposition 5.4 assures us that the vanishing of mixed cumulants $\kappa_n^{(cc)}$ is equivalent to the one of mixed cumulants $\kappa_n^{(c)}$.

(2) $\underline{\kappa}^{(cc)}$ is useful because it can be related to the lattice operations on $NC^{(B-opp)}(n)$. This allows, for instance, a nicely streamlined treatment of the formula for cumulants with products as entries (as shown in Theorem 2.6 of [8]).

(3) The formula (5.2) which was used for introducing $\underline{\kappa}^{(cc)}$ in Definition 5.2 can be re-written in a way which only uses partitions with zero-blocks on the right-hand side. Indeed, it is immediate that the sub-sum over $\{\sigma \in NC^{(B-opp)}(n) : \sigma \text{ has no zero-blocks}\}$ on the right-hand side of (5.2) simply gives $\varphi_n(x_1, \ldots, x_n)$. With this observation, Equation (5.2) can be put in the form

$$(\chi_n - \varphi_n)(x_1, \ldots, x_n) = \sum_{\substack{\sigma \in NC^{(B-opp)}(n) \\ with\ zero-blocks}} \prod_{\substack{Z \in \sigma, \\ Z=-Z}} \kappa^{(cc)}_{|Z|/2}(\,(x_1, \ldots, x_n) \mid \mathrm{Abs}(Z)) \times \qquad (5.8)$$
$$\times \prod_{\substack{pairs\, U,-U \in \sigma, \\ such\ that\, U \neq -U}} \kappa_{|U|;\underline{\varphi}}(\,(x_1, \ldots, x_n) \mid \mathrm{Abs}(U));$$

this is strikingly similar to the formula (5.1) which was reviewed in Section 5.1 in the framework of infinitesimal free cumulants.

The apparent resemblance between Equations (5.1) and (5.8) is, in some sense, a manifestation of the resemblance between the two types of pictures shown in Figs. 2 and 3, for partitions in $NC^{(B)}(n)$ and in $NC^{(B-opp)}(n)$. We should point out here the intriguing additional detail that the lattices $NC^{(B)}(n)$ and $NC^{(B-opp)}(n)$ have the same cardinality – they are both counted by the binomial coefficient $2n$-choose-n. Tantalizing as this may be, we are not aware of any result that would relate c-free independence to infinitesimal free independence via a direct connection between $NC^{(B)}(n)$ and $NC^{(B-opp)}(n)$; an example of a rather natural bijection $NC^{(B)}(n) \to NC^{(B-opp)}(n)$, based on systems of parentheses, was explained to one of us by Vic Reiner [18], but that does not seem to convert well into considerations on non-commutative random variables.

6. Proof of Theorem 1.7

Notation 6.1. *(Framework of the section.)*

Throughout this section we fix a vector space $\mathcal{V}$ over $\mathbb{C}$ and two families of multilinear functionals $\underline{\varphi}, \underline{\chi} \in \mathfrak{M}_{\mathcal{V}}$, where $\underline{\varphi} = (\varphi_n)_{n=1}^{\infty}$ and $\underline{\chi} = (\chi_n)_{n=1}^{\infty}$, as in Notation 1.5. We assume that $\underline{\varphi}$ is tracial, in the sense indicated in that same notation. We also fix a linear map $\Delta : \mathcal{V} \to \mathcal{V} \otimes \mathcal{V}$, and we consider the transformation $\Delta^* : \mathfrak{M}_{\mathcal{V}} \to \mathfrak{M}_{\mathcal{V}}$ constructed from Δ in the way described in Notation 1.6. We then put $\underline{\varphi}' := \Delta^*\left(\underline{\beta}_{\underline{\chi}}\right)$, where $\underline{\beta}_{\underline{\chi}} = (\beta_{n;\underline{\chi}})_{n=1}^{\infty} \in \mathfrak{M}_{\mathcal{V}}$ is the family of Boolean cumulants of $\underline{\chi}$. Our goal for the section is to prove the equality claimed in Theorem 1.7:

$$\underline{\kappa}' = \Delta^*(\,\underline{\kappa}^{(c)}\,),$$

where $\underline{\kappa}' = (\kappa_n')_{n=1}^{\infty} \in \mathfrak{M}_{\mathcal{V}}$ are the infinitesimal free cumulants associated to $(\mathcal{V}, \underline{\varphi}, \underline{\varphi}')$, while $\underline{\kappa}^{(c)} = (\kappa_n^{(c)})_{n=1}^{\infty}$ are the c-free cumulants associated to $(\mathcal{V}, \underline{\varphi}, \underline{\chi})$. In view of how Δ^* is defined, proving the latter equality amounts to proving that for every $n \in \mathbb{N}$ one has

$$\kappa_n' = \sum_{m=1}^{n} \kappa_{n+1}^{(c)} \circ \Gamma_{n+1}^{m} \circ \Delta_n^{(m)}, \tag{6.1}$$

with $\Gamma_{n+1} : \mathcal{V}^{\otimes(n+1)} \to \mathcal{V}^{\otimes(n+1)}$ and $\Delta_n^{(m)} : \mathcal{V}^{\otimes n} \to \mathcal{V}^{\otimes(n+1)}$ as in (1) and (2) of Notation 1.6. In Equation (6.1), κ_n' and $\kappa_{n+1}^{(c)}$ are treated as linear functionals on $\mathcal{V}^{\otimes n}$ and respectively $\mathcal{V}^{\otimes(n+1)}$ (rather than multilinear functionals on $\mathcal{V}^n$ and $\mathcal{V}^{n+1}$).

In order to handle the left-hand side of Equation (6.1), it is convenient to introduce the following notation.

Notation 6.2. (1) For every $n \in \mathbb{N}$ and $m \in \{1, \ldots, n\}$, we denote

$$\gamma_n^{(m)} := \beta_{n+1;\underline{\chi}} \circ \Gamma_{n+1}^{m} \circ \Delta_n^{(m)}. \tag{6.2}$$

We will view $\gamma_n^{(m)}$, as needed, either as a linear functional $\mathcal{V}^{\otimes n} \to \mathbb{C}$ or as a multilinear functional $\mathcal{V}^n \to \mathbb{C}$.

(2) For every $n \in \mathbb{N}$, $m \in \{1, \ldots, n\}$ and $\pi \in NC(n)$ we will denote by $\gamma_\pi^{(m)}$ the multilinear functional $\mathcal{V}^n \to \mathbb{C}$ defined as follows:

$$\gamma_\pi^{(m)}(x_1, \ldots, x_n) := \gamma_{|V_o|}^{(r)}((x_1, \ldots, x_n) \mid V_o) \cdot \prod_{\substack{V \in \pi \\ V \neq V_o}} \varphi_{|V|}((x_1, \ldots, x_n) \mid V), \tag{6.3}$$

where V_o denotes the block of π which contains the number m, and r denotes the "rank of m inside V_o" – that is, upon writing $V_o = \{i_1, \ldots, i_p\}$ with $i_1 < \cdots < i_p$, one has $m = i_r$.

[Note that part (2) of this notation is an extension of part (1), since $\gamma_n^{(m)}$ may be retrieved as $\gamma_\pi^{(m)}$ for $\pi = 1_n \in NC(n)$.]

Lemma 6.3. *For every $n \in \mathbb{N}$, one has*

$$\kappa_n' = \sum_{\pi \in NC(n)} \sum_{m=1}^{n} M\ddot{o}b_n(\pi, 1_n)\, \gamma_\pi^{(m)}. \tag{6.4}$$

Proof. We start from the formula (3.7) which defines $\kappa_n'(x_1, \ldots, x_n)$, and on the right-hand side of that formula we replace the quantity $\varphi_{|V_o|}'\big((x_1, \ldots, x_n) \mid V_o\big)$ by using the definition of $\varphi_{|V_o|}'$, which is

$$\varphi'_{|V_o|} = \beta_{|V_o|+1;\underline{\chi}} \circ \widetilde{\Delta}_{|V_o|} = \sum_{j=1}^{|V_o|} \beta_{|V_o|+1;\underline{\chi}} \circ \Gamma^{j}_{|V_o|+1;\underline{\chi}} \circ \Delta^{(j)}_{|V_o|}.$$

It is convenient to re-write the latter sum in the equivalent form

$$\sum_{m \in V_o} \beta_{|V_o|+1;\underline{\chi}} \circ \Gamma^{r(m)}_{|V_o|+1;\underline{\chi}} \circ \Delta^{(r(m))}_{|V_o|}, \tag{6.5}$$

where $r(m)$ indicates the rank of m within the block V_o (as indicated in Notation 6.2(2)). Upon substituting (6.5) in (3.7), one arrives to an equation of the form

$$\kappa'_n(x_1, \dots, x_n) = \sum_{\pi \in NC(n)} \sum_{V_o \in \pi} \sum_{m \in V_o} \text{Möb}_n(\pi, 1_n) \cdot \text{term}(\pi, V_o, m), \tag{6.6}$$

and the reader should have no difficulty to verify that the quantity "term(π, V_o, m)" appearing in (6.6) is nothing but the $\gamma_\pi^{(m)}(x_1, \dots, x_n)$ from Notation 6.2(2). Finally, the double sum $\sum_{V_o \in \pi} \sum_{m \in V_o}$ in (6.6) can be re-written as a plain sum $\sum_{m=1}^{n}$, which leads to the formula (6.4) stated in the lemma. □

We next introduce a notation which captures the multilinear functionals used for the description of c-free cumulants in Proposition 4.1.

Notation 6.4. For $n \in \mathbb{N}$ and $\pi \in NC(n)$ such that $\pi \ll 1_n$ we define $\eta_\pi : \mathcal{V}^n \to \mathbb{C}$ by

$$\eta_\pi(x_1, \dots, x_n) = \beta_{|W_o|;\underline{\chi}}\left((x_1, \dots, x_n)|W_o\right) \cdot \prod_{\substack{V \in \pi, \\ V \neq W_o}} \varphi_{|V|}\left((x_1, \dots, x_n)|V\right),$$

where W_o is the block of π which contains the numbers 1 and n.

Lemma 6.5. *For every $n \in \mathbb{N}$, the right-hand side of Equation (6.1) can be written as*

$$\sum_{\substack{\rho \in NC(n+1), \\ \rho \ll 1_{n+1}}} \sum_{m=1}^{n} \textit{Möb}_n(\rho, 1_{n+1})\ \eta_\rho \circ \Gamma^{(m)}_{n+1} \circ \Delta^{(m)}_n. \tag{6.7}$$

Proof. Proposition 4.1 tells us that

$$\kappa^{(c)}_{n+1} = \sum_{\substack{\rho \in NC(n+1), \\ \rho \ll 1_{n+1}}} \text{Möb}_n(\rho, 1_{n+1})\ \eta_\rho.$$

Substituting this on the right-hand side of Equation (6.1) leads to (6.7). □

Returning to the formula (6.1) that needs to be proved: we now have both its sides expressed as sums, in Lemmas 6.3 and 6.5. Our proof of (6.1) will go by showing that the sums appearing in the two said lemmas can be identified term by term. In order to do the identification of the indexing sets, we will use the bijection described as follows.

Remark and Notation 6.6. Let $n \in \mathbb{N}$ and $m \in \{1, \ldots, n\}$ be given. We consider the natural bijections

$$\{\rho \in NC(n+1) : \rho \ll 1_{n+1}\} \to \left\{ \widehat{\rho} \in NC(n+1) \,\middle|\, \begin{array}{c} m \text{ and } m+1 \\ \text{belong to the} \\ \text{same block of } \widehat{\rho} \end{array} \right\} \to NC(n), \tag{6.8}$$

where:

– The first map in (6.8) does a forward cyclic translation by m; that is, this map sends ρ to $\widehat{\rho} = \{\tau_m(V) : V \in \rho\}$, with $\tau_m(k) = m + k \bmod (n+1)$, $1 \leq k \leq n+1$.

– The second map in (6.8) merges together the numbers m and $m+1$ in the block of $\widehat{\rho}$ which contains them.

We will use the notation

$$F_n^{(m)} : \{\rho \in NC(n+1) : \rho \ll 1_{n+1}\} \to NC(n)$$

for the bijection obtained by composing the two maps from (6.8).

It is useful to observe that for every $\rho \in NC(n+1)$ such that $\rho \ll 1_{n+1}$, one has

$$\text{Möb}_n\big(F_n^{(m)}(\rho), 1_n\big) = \text{Möb}_{n+1}(\rho, 1_{n+1}). \tag{6.9}$$

Indeed, if we follow the arrows $\rho \mapsto \widehat{\rho} \mapsto \pi = F_n^{(m)}(\rho)$ in (6.8), one first has that $\text{Möb}_{n+1}(\rho, 1_{n+1}) = \text{Möb}_{n+1}(\widehat{\rho}, 1_{n+1})$, because the cyclic translation by m gives an automorphism of $NC(n+1)$ which preserves the values of the Möbius function. Then one has the equality $\text{Möb}_{n+1}(\widehat{\rho}, 1_{n+1}) = \text{Möb}_n(\pi, 1_n)$, which is seen by writing the explicit formulas of $\text{Möb}_{n+1}(\widehat{\rho}, 1_{n+1})$ and of $\text{Möb}_n(\pi, 1_n)$ (as in Remark 2.4), and by observing that the block structure of $K_{n+1}(\widehat{\rho})$ only differs from the one of $K_n(\pi)$ by a singleton-block at m.

Lemma 6.7. *Let $n \in \mathbb{N}$ and $m \in \{1, \ldots, n\}$ be given. Let us also fix a partition $\rho \in NC(n+1)$ such that $\rho \ll 1_{n+1}$, and let us denote $\pi := F_n^{(m)}(\rho) \in NC(n)$. We then have*

$$\gamma_\pi^{(m)} = \eta_\rho \circ \Gamma_{n+1}^m \circ \Delta_n^{(m)} \tag{6.10}$$

(equality of multilinear functionals on $\mathcal{V}^n$).

Proof. The verification of Equation (6.10) is done by a mere unfolding of the definitions of the functionals indicated on the two sides of the equation. As this unfolding only

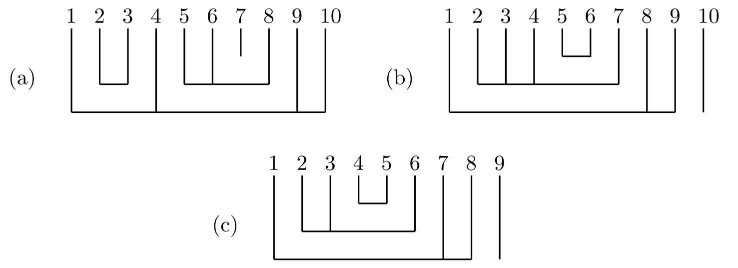


Fig. 4. (a) The partition $\rho \ll 1_{10}$ used for illustration in this proof. (b) The cyclic permutation $\widehat{\rho}$ of ρ by $m = 3$. (c) The partition $\pi = F_9^{(3)}(\rho) \in NC(9)$.

presents difficulties of notational nature, we believe it is more beneficial to the reader if we show it on a suitable concrete example which captures the relevant features of what is going on. At the end of the proof we will elaborate on why the details of the verification are in fact general, rather than being specific to the example shown.

The concrete example we pick for illustration has

$$n = 9,\ m = 3,\ \rho = \big\{ \{1,4,9,10\},\ \{2,3\},\ \{5,6,8\},\ \{7\} \big\} \in NC(10).$$

Fig. 4 above depicts the partition ρ, the partition

$$\pi = F_9^{(3)}(\rho) = \big\{ \{1,7,8\},\ \{2,3,6\},\ \{4,5\},\ \{9\} \big\} \in NC(9),$$

and also the partition $\widehat{\rho} \in NC(10)$ which is used as an intermediate between ρ and π in Notation 6.6.

Given a tuple $(x_1, \ldots, x_n) \in \mathcal{V}^n$, the processing of either side of Equation (6.10) will require the explicit writing of $\Delta(x_m) \in \mathcal{V} \otimes \mathcal{V}$. We will denote

$$\Delta(x_m) = \sum_{h=1}^{s} y_h \otimes z_h.$$

In the concrete example used for illustration (with $n = 9, m = 3$), the block of π which contains m is $V_o = \{2,3,6\}$, and the rank of m in V_o is $r = 2$; hence the evaluation of the left-hand side of Equation (6.10) starts with

$$\gamma_\pi^{(3)}(x_1, \ldots, x_9) = \gamma_3^{(2)}(x_2, x_3, x_6) \cdot \varphi_3(x_1, x_7, x_8)\varphi_2(x_4, x_5)\varphi_1(x_9), \tag{6.11}$$

where we then replace

$$\begin{aligned}\gamma_3^{(2)}(x_2, x_3, x_6) &= (\beta_{4;\underline{\chi}} \circ \Gamma_4^2 \circ \Delta_3^{(2)})(x_2 \otimes x_3 \otimes x_6) \text{ (cf. Notation 6.2(1))} \\ &= (\beta_{4;\underline{\chi}} \circ \Gamma_4^2)(x_2 \otimes \Delta(x_3) \otimes x_6) \\ &= \sum_{h=1}^{s} \beta_{4;\underline{\chi}}\big(\Gamma_4^2(x_2 \otimes y_h \otimes z_h \otimes x_6)\big)\end{aligned}$$

$$= \sum_{h=1}^{s} \beta_{4;\underline{\chi}}(z_h \otimes x_6 \otimes x_2 \otimes y_h) = \sum_{h=1}^{s} \beta_{4;\underline{\chi}}(z_h, x_6, x_2, y_h)$$

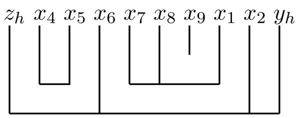
(with $\beta_{4;\underline{\chi}}$ interchangeably viewed as a linear functional on $\mathcal{V}^{\otimes 4}$ or as a multilinear functional on $\mathcal{V}^4$). Hence the evaluation started in (6.11) comes to

$$\gamma_\pi^{(3)}(x_1, \ldots, x_9) = \sum_{h=1}^{s} \beta_{4;\underline{\chi}}(z_h, x_6, x_2, y_h) \cdot \varphi_3(x_1, x_7, x_8)\varphi_2(x_4, x_5)\varphi_1(x_9). \tag{6.12}$$

Moving now to the corresponding evaluation on the right-hand side of Equation (6.10), we have

$$(\eta_\rho \circ \Gamma_{10}^3 \circ \Delta_9^{(3)})(x_1, \ldots, x_9) = (\eta_\rho \circ \Gamma_{10}^3)(x_1 \otimes x_2 \otimes \Delta(x_3) \otimes x_4 \otimes \cdots \otimes x_9)$$
$$= \sum_{h=1}^{s} \eta_\rho\big(\Gamma_{10}^3(x_1 \otimes x_2 \otimes y_h \otimes z_h \otimes x_4 \otimes \cdots \otimes x_9)\big)$$
$$= \sum_{h=1}^{s} \eta_\rho(z_h \otimes x_4 \otimes \cdots \otimes x_9 \otimes x_1 \otimes x_2 \otimes y_h) = \sum_{h=1}^{s} \eta_\rho(z_h, x_4, \ldots, x_9, x_1, x_2, y_h).$$

In the latter expression, we use the definition of η_ρ from Notation 6.4; when doing so, it is instructive to re-draw the picture of ρ from Fig. 4, with the labels $1, 2, \ldots, 10$ being replaced by labels "$z_h, x_4, \ldots, x_2, y_h$":



The result of the evaluation for the right-hand side of Equation (6.10) then comes to

$$(\eta_\rho \circ \Gamma_{10}^3 \circ \Delta_9^{(3)})(x_1, \ldots, x_9) \tag{6.13}$$
$$= \sum_{h=1}^{s} \beta_{4;\underline{\chi}}(z_h, x_6, x_2, y_h) \cdot \varphi_2(x_4, x_5)\varphi_3(x_7, x_8, x_1)\varphi_1(x_9).$$

The expression obtained on the right-hand side of Equation (6.13) coincides with the one on the right-hand side of Equation (6.12), modulo the detail that the arguments x_1, x_7, x_8 were cyclically rotated in (6.13) (which doesn't change, however, the value of $\varphi_3(x_1, x_7, x_8)$, due to the hypothesis that $\underline{\varphi}$ is tracial). This completes the verification of Equation (6.10) in the concrete example picked for illustration.

We conclude our argument with a discussion about what is the general structure of the expressions encountered in (6.12) and (6.13), and what were the relations between the block structure of π and of ρ which produced the equality between these expressions. We hope that, upon examination, the reader will agree that this discussion is not specific to the special example considered above for illustration, but can be made whenever we consider partitions $\rho \in NC(n+1)$ and $\pi = F_n^{(m)}(\rho) \in NC(n)$ as described in the statement of the lemma.

– We first note that the processing of either side of Equation (6.10) leads to an expression which is a sum of products. The number s of terms in the sum is picked from an explicit writing of $\Delta(x_m)$, and every term has one factor which is a Boolean cumulant of $\underline{\chi}$, multiplied by several factors which are moments of $\underline{\varphi}$.

– The factors which are Boolean cumulants of $\underline{\chi}$ are found by looking at the block $V_o = \{i_1 < \cdots < i_p\}$ of π which contains the number m and, on the other hand, at the unique outer block W_o of ρ. The sets V_o and W_o are related: one obtains V_o out of W_o via a cyclic rotation by m followed by merging of m with $m+1$, as explained in Notation 6.6. Both the processing using V_o in (6.12) and the one using W_o in (6.13) involve the same Boolean cumulants, which are[4] of the form

$$\beta_{p+1;\underline{\chi}}(z_h, x_{i_{r+1}}, \ldots, x_{i_p}, x_{i_1}, \ldots, x_{i_{r-1}}, y_h), \text{ with } 1 \leq h \leq s.$$

– The factors which are moments of $\underline{\varphi}$ are found by looking at blocks $V \neq V_o$ of π (for (6.12)) and at blocks $W \neq W_o$ of ρ (for (6.13)). One has a natural bijective correspondence between such V's and W's, where every V is obtained from the corresponding W by a cyclic permutation followed by suitable relabeling. (For instance, in the example of Fig. 4: the block $W = \{2, 3\}$ of ρ is cyclically permuted to $\{5, 6\}$ and then relabeled to become the block $V = \{4, 5\}$ of π.) For a V and W that correspond to each other: the definitions of the functionals $\gamma_\pi^{(m)}$ and η_ρ are made in such a way that the choice of components selected out of a tuple $(x_1, \ldots, x_n) \in \mathcal{V}^n$ is the same when we look at V in (6.12) and when we look at W in (6.13), modulo a possible cyclic permutation of the arguments. This possible cyclic permutation of the arguments is, however, taken care by the hypothesis that $\underline{\varphi}$ is tracial. □

6.8. Conclusion of the proof of Theorem 1.7. In view of Lemmas 6.3 and 6.5, we are left to show that, for every $n \in \mathbb{N}$, one has

$$\sum_{\pi \in NC(n)} \sum_{m=1}^{n} \text{Möb}_n(\pi, 1_n)\, \gamma_\pi^{(m)} = \sum_{\substack{\rho \in NC(n+1), \\ \rho \ll 1_{n+1}}} \sum_{m=1}^{n} \text{Möb}_{n+1}(\rho, 1_{n+1})\, \eta_\rho \circ \Gamma_{n+1}^{(m)} \circ \Delta_n^{(m)}. \tag{6.14}$$

And indeed, for any fixed $n \in \mathbb{N}$ and $m \in \{1, \ldots, n\}$ we see that

[4] It is fortunate that for these factors we don't need to do any kind of permutation in the arguments of the Boolean cumulants that appear. Indeed, Boolean cumulants are not invariant under cyclic permutations of variables – so even if we required $\underline{\chi}$ to be tracial, this feature would not be passed onto $\underline{\beta}_{\underline{\chi}}$.

$$\sum_{\pi \in NC(n)} \text{Möb}_n(\pi, 1_n)\, \gamma_\pi^{(m)} = \sum_{\substack{\rho \in NC(n+1), \\ \rho \ll 1_{n+1}}} \text{Möb}_n(F_n^{(m)}(\rho), 1_n)\ \gamma^{(m)}_{F_n^{(m)}(\rho)}$$

(via the change of summation variable "$\pi = F_n^{(m)}(\rho)$", based on Remark 6.6), which can be continued with

$$= \sum_{\substack{\rho \in NC(n+1), \\ \rho \ll 1_{n+1}}} \text{Möb}_{n+1}(\rho, 1_{n+1})\ \eta_\rho \circ \Gamma_{n+1}^{(m)} \circ \Delta_n^{(m)}$$

(by Lemma 6.7 and the equality of Möbius functions observed in (6.9)). Summing over $m \in \{1, \ldots, n\}$ in the latter equalities leads to (6.14). $\square$

7. Proof of Theorem 1.4

In this section we re-connect with the framework of Section 1.2 of the Introduction, and we explain how Theorem 1.4 is a consequence of Theorem 1.7.

Notation 7.1. We fix throughout the section a positive integer k. We put $\mathcal{V} := \mathbb{C}^k$, we fix a basis $e_1, \ldots, e_k$ of $\mathcal{V}$, and we consider the linear map $\Delta : \mathcal{V} \to \mathcal{V} \otimes \mathcal{V}$ determined by the requirement that

$$\Delta(e_i) = e_i \otimes e_i, \quad 1 \leq i \leq k. \tag{7.1}$$

We then consider the space $\mathfrak{M}_\mathcal{V}$ of families of multilinear functionals defined as in Notation 1.5, and the transformation $\Delta^* : \mathfrak{M}_\mathcal{V} \to \mathfrak{M}_\mathcal{V}$ constructed by starting from Δ in the way shown in Notation 1.6. It is immediate that, in the special case considered here, Δ^* acts as follows: given $\underline{\varphi} = (\varphi_n)_{n=1}^\infty \in \mathfrak{M}_\mathcal{V}$, one has $\Delta^*(\underline{\varphi}) = \underline{\psi} = (\psi_n)_{n=1}^\infty$, with ψ_n determined via the requirement that for every $i_1, \ldots i_n \in \{1, \ldots, k\}$ one has

$$\psi_n(e_{i_1}, \ldots, e_{i_n}) = \sum_{m=1}^{n} \varphi_{n+1}(e_{i_m}, \ldots, e_{i_n}, e_{i_1}, \ldots, e_{i_m}). \tag{7.2}$$

Remark and Notation 7.2. In the present section, the notations for families of cumulants from Section 3 get to be used in two distinct ways. On the one hand, we can consider families of functionals $\underline{\varphi}, \underline{\chi}, \underline{\varphi}' \in \mathfrak{M}_\mathcal{V}$; in connection to such families we can define the four brands of cumulants discussed in Section 3, which will be denoted here with appropriate indices such as

$$\begin{cases} \underline{\kappa}_{\underline{\varphi}} = (\,\kappa_{n;\underline{\varphi}}\,)_{n=1}^{\infty}, \text{ for the free cumulants of } \underline{\varphi}; \\ \underline{\beta}_{\underline{\chi}} = (\,\beta_{n;\underline{\chi}}\,)_{n=1}^{\infty}, \text{ for the Boolean cumulants of } \underline{\chi}; \\ \underline{\kappa}^{(c)}_{(\underline{\varphi},\underline{\chi})} = (\,\kappa^{(c)}_{n;(\underline{\varphi},\underline{\chi})}\,)_{n=1}^{\infty}, \text{ for the c-free cumulants of } \underline{\varphi} \text{ and } \underline{\chi}; \\ \underline{\kappa}'_{(\underline{\varphi},\underline{\varphi}')} = (\,\kappa'_{n;(\underline{\varphi},\underline{\varphi}')}\,)_{n=1}^{\infty}, \text{ for the infinitesimal free cumulants of } \underline{\varphi} \text{ and } \underline{\varphi}'. \end{cases} \tag{7.3}$$

On the other hand, we can consider linear functionals μ, ν, μ' on the algebra of non-commutative polynomials $\mathbb{C}\langle X_1, \ldots, X_k\rangle$, with $\mu(1) = \nu(1) = 1$ and $\mu'(1) = 0$. In connection to such functionals: since $\mathbb{C}\langle X_1, \ldots, X_k\rangle$ is a unital algebra (where a linear functional naturally induces multilinear functionals on all the powers $\mathbb{C}\langle X_1, \ldots, X_k\rangle^n$) we can also define families of cumulants, with notations such as

$$\begin{cases} \underline{\kappa}_{\mu} = (\,\kappa_{n;\mu}\,)_{n=1}^{\infty}, \text{ for the free cumulants of } \mu; \\ \underline{\beta}_{\nu} = (\,\beta_{n;\nu}\,)_{n=1}^{\infty}, \text{ for the Boolean cumulants of } \nu; \\ \underline{\kappa}^{(c)}_{(\mu,\nu)} = (\,\kappa^{(c)}_{n;(\mu,\nu)}\,)_{n=1}^{\infty}, \text{ for the c-free cumulants of } \mu \text{ and } \nu; \\ \underline{\kappa}'_{(\mu,\mu')} = (\,\kappa'_{n;(\mu,\mu')}\,)_{n=1}^{\infty}, \text{ for the infinitesimal free cumulants of } \mu \text{ and } \mu'. \end{cases} \tag{7.4}$$

One has an obvious connection between the families of cumulants listed in (7.3) and in (7.4), as recorded in the next lemma.

Lemma 7.3. *Let $\mu, \nu, \mu' : \mathbb{C}\langle X_1, \ldots, X_k\rangle \to \mathbb{C}$ be linear functionals with $\mu(1) = \nu(1) = 1$ and $\mu'(1) = 0$. Consider the families of functionals $\underline{\varphi}, \underline{\chi}$ and $\underline{\varphi}'$ in $\mathfrak{M}_{\mathcal{V}}$ defined via the requirement that for every $n \in \mathbb{N}$ and $i_1, \ldots, i_n \in \{1, \ldots, k\}$ we have*

$$\begin{cases} \varphi_n(e_{i_1}, \ldots, e_{i_n}) & = \mu(X_{i_1} \cdots X_{i_n}), \\ \chi_n(e_{i_1}, \ldots, e_{i_n}) & = \nu(X_{i_1} \cdots X_{i_n}), \text{ and} \\ \varphi'_n(e_{i_1}, \ldots, e_{i_n}) & = \mu'(X_{i_1} \cdots X_{i_n}). \end{cases} \tag{7.5}$$

Then, for every $n \in \mathbb{N}$ and $i_1, \ldots, i_n \in \{1, \ldots, k\}$, we have the following equalities of cumulants:

(1) $\kappa_{n;\mu}(X_{i_1}, \ldots, X_{i_n}) = \kappa_{n;\underline{\varphi}}(e_{i_1}, \ldots, e_{i_n})$.
(2) $\beta_{n;\nu}(X_{i_1}, \ldots, X_{i_n}) = \beta_{n;\underline{\chi}}(e_{i_1}, \ldots, e_{i_n})$.
(3) $\kappa^{(c)}_{n;(\mu,\nu)}(X_{i_1}, \ldots, X_{i_n}) = \kappa^{(c)}_{n;(\underline{\varphi},\underline{\chi})}(e_{i_1}, \ldots, e_{i_n})$.
(4) $\kappa'_{n;(\mu,\mu')}(X_{i_1}, \ldots, X_{i_n}) = \kappa'_{n;(\underline{\varphi},\underline{\varphi}')}(e_{i_1}, \ldots, e_{i_n})$.

Proof. This is because, in each of the equations listed in the conclusion of the lemma, the same moment-cumulant formulas are used on both sides of the equation. □

Lemma 7.4. *Consider the framework and notations of Lemma 7.3. Suppose we have $\mu' = \Psi_k(\nu)$, in the sense of Definition 1.1. Then it follows that $\underline{\varphi}' = \Delta^*(\underline{\beta}_{\underline{\chi}})$, with Δ^* as reviewed in Notation 7.1.*

Proof. Let $\Delta^*(\underline{\beta_\chi}) := \underline{\psi} = (\psi_n)_{n=1}^\infty$. For every $n \in \mathbb{N}$ and $i_1, \ldots, i_n \in \{1, \ldots, k\}$ we write

$$\begin{aligned}\psi_n(e_{i_1}, \ldots, e_{i_n}) &= \sum_{m=1}^n \beta_{n+1;\underline{\chi}}(e_{i_m}, \ldots, e_{i_n}, e_{i_1}, \ldots, e_{i_m}) \text{ (by Equation (7.2))}\\ &= \sum_{m=1}^n \beta_{n+1;\nu}(X_{i_m}, \ldots, X_{i_n}, X_{i_1}, \ldots, X_{i_m}) \text{ (by Lemma 7.3(2))}\\ &= \mu'(X_{i_1} \cdots X_{i_n}) \text{ (by Equation (1.6) in Definition 1.1)}\\ &= \varphi'_n(e_{i_1}, \ldots, e_{i_n}) \text{ (by the definition of } \varphi'_n \text{ in Eqn. (7.5)).}\end{aligned}$$

By multilinearity, it follows that $\psi_n = \varphi'_n$ for all $n \in \mathbb{N}$. Hence $\underline{\varphi}' = \underline{\psi} = \Delta^*(\underline{\beta_\chi})$. □

7.5. Proof of Theorem 1.4. Recall that we have the following data: k is a positive integer, μ, ν are in $\mathcal{D}_{\text{alg}}(k)$, and we let $\mu' := \Psi_k(\nu) \in \mathcal{D}'_{\text{alg}}(k)$. We assume μ to be tracial. We have to prove that the infinitesimal free cumulants of (μ, μ') are related to the c-free cumulants of (μ, ν) by the formula

$$\kappa'_{n;(\mu,\mu')}(X_{i_1}, \ldots, X_{i_n}) = \sum_{m=1}^n \kappa^{(c)}_{n+1;(\mu,\nu)}(X_{i_m}, \ldots, X_{i_n}, X_{i_1}, \ldots, X_{i_m}),$$

holding for every $n \in \mathbb{N}$ and $i_1, \ldots, i_n \in \{1, \ldots, k\}$.

Consider the families of functionals $\underline{\varphi}, \underline{\chi}, \underline{\varphi}' \in \mathfrak{M}_\mathcal{V}$ defined as in Lemma 7.3. Note that the hypothesis of μ being tracial immediately implies that $\underline{\varphi}$ is tracial as well. On the other hand, Lemma 7.4 gives us the fact that $\underline{\varphi}' = \Delta^*(\underline{\beta_\chi})$. Theorem 1.7 then applies to $\underline{\varphi}, \underline{\chi}, \underline{\varphi}'$, which leads to a formula relating the infinitesimal free cumulants $\kappa_{n;(\underline{\varphi},\underline{\varphi}')}$ to the c-free cumulants $\kappa^{(c)}_{n;(\underline{\varphi},\underline{\chi})}$. We are only left to write that, for every $n \in \mathbb{N}$ and $i_1, \ldots, i_n \in \{1, \ldots, k\}$:

$$\begin{aligned}\kappa'_{n;(\mu,\mu')}(X_{i_1}, \ldots, X_{i_n}) &= \kappa'_{n;(\underline{\varphi},\underline{\varphi}')}(e_{i_1}, \ldots, e_{i_n}) \text{ (by Lemma 7.3(4))}\\ &= \sum_{m=1}^n \kappa^{(c)}_{n+1;(\underline{\varphi},\underline{\chi})}(e_{i_m}, \ldots, e_{i_n}, e_{i_1}, \ldots, e_{i_m}) \text{ (by Theorem 1.7)}\\ &= \sum_{m=1}^n \kappa^{(c)}_{n+1;(\mu,\nu)}(X_{i_m}, \ldots, X_{i_n}, X_{i_1}, \ldots, X_{i_m}),\end{aligned}$$

where at the latter equality we used Lemma 7.3(3). □

8. c-free and infinitesimally free products and additive convolutions

In this section we continue with the framework of algebraic distributions in $\mathcal{D}_{\text{alg}}(k)$ and $\mathcal{D}'_{\text{alg}}(k)$, and with the various families of cumulants associated to such distributions, with notations as in (7.4) of the preceding section. We will review the variations of the notions of free product and free additive convolution that are relevant for the present paper, and we will show how Theorems 1.2 and 1.3 follow from the formula about cumulants obtained in Theorem 1.4. Since the latter formula only addresses cumulants for tuples of the special form $(X_{i_1}, \ldots, X_{i_n})$ (rather than covering cumulants for tuples $(P_1, \ldots, P_n)$ with general $P_1, \ldots, P_n \in \mathbb{C}\langle X_1, \ldots, X_k \rangle$) it is useful to start by recording the following lemma.

Lemma 8.1. *Let k be a positive integer.*

(1) Let μ_1, μ_2 be in $\mathcal{D}_{\text{alg}}(k)$, and suppose that the free cumulant functionals of μ_1 agree with those of μ_2 on all tuples $\{(X_{i_1}, \ldots, X_{i_n}) : n \in \mathbb{N}, 1 \leq i_1, \ldots, i_n \leq k\}$. Then $\mu_1 = \mu_2$.

(2) Let μ, ν_1, ν_2 be in $\mathcal{D}_{\text{alg}}(k)$, and suppose that the c-free cumulants of (μ, ν_1) agree with those of (μ, ν_2) on all tuples $\{(X_{i_1}, \ldots, X_{i_n}) : n \in \mathbb{N}, 1 \leq i_1, \ldots, i_n \leq k\}$. Then $\nu_1 = \nu_2$.

(3) Let $\mu \in \mathcal{D}_{\text{alg}}(k)$ and $\mu'_1, \mu'_2 \in \mathcal{D}'_{\text{alg}}(k)$, and suppose the infinitesimal free cumulants of (μ, μ'_1) agree with those of (μ, μ'_2) on all tuples $\{(X_{i_1}, \ldots, X_{i_n}) : n \in \mathbb{N}, 1 \leq i_1, \ldots, i_n \leq k\}$. Then $\mu'_1 = \mu'_2$.

Proof. Each of the three parts of the lemma follows by an immediate application of the suitable moment-cumulant formula. For instance: in the case of statement (3), the formula (3.6) used in the definition of infinitesimal free cumulants gives us that both $\mu'_1(X_{i_1} \cdots X_{i_n})$ and $\mu'_2(X_{i_1} \cdots X_{i_n})$ are equal to

$$\sum_{\pi \in NC(n)} \sum_{V_o \in \pi} \kappa'_{|V_o|}\big((X_{i_1}, \ldots, X_{i_n}) \mid V_o\big) \cdot \prod_{\substack{V \in \pi \\ V \neq V_o}} \kappa_{|V|}\big((X_{i_1}, \ldots, X_{i_n}) \mid V\big),$$

where the $\kappa_{|V|}$s are free cumulant of μ, while $\kappa'_{|V_o|}\big((X_{i_1}, \ldots, X_{i_n}) \mid V_o\big)$ is the common value of the infinitesimal free cumulants of (μ, μ'_1) and (μ, μ'_2) on the tuple $(X_{i_1}, \ldots, X_{i_n}) \mid V_o$. Thus μ'_1 and μ'_2 agree on all monomials $X_{i_1} \cdots X_{i_n}$ with $n \in \mathbb{N}$ and $i_1, \ldots, i_n \in \{1, \ldots, k\}$, and it follows that $\mu'_1 = \mu'_2$, as required. □

8.1. Free product operations and the proof of Theorem 1.3

In this subsection we review the necessary extensions of the notion of free product of distributions, and we show how Theorem 1.3 follows from Theorem 1.4.

For our purposes, it is most convenient to treat all versions of free products in terms of cumulants. This starts with the standard free product operation, as described for instance in Lecture 6 of [15]. Indeed, the free product $\mu_1 \star \mu_2$ of two distributions $\mu_1 \in \mathcal{D}_{\text{alg}}(k)$ and $\mu_2 \in \mathcal{D}_{\text{alg}}(\ell)$ (for some $k, \ell \in \mathbb{N}$) can be identified as the distribution $\widetilde{\mu} \in \mathcal{D}_{\text{alg}}(k+\ell)$ which is uniquely determined by the requirement that its free cumulants fulfill the following condition: for every $n \in \mathbb{N}$ and $i_1, \ldots, i_n \in \{1, \ldots, k+\ell\}$, one has

$$\kappa_{n;\widetilde{\mu}}(X_{i_1}, \ldots, X_{i_n}) = \begin{cases} \kappa_{n;\mu_1}(X_{i_1}, \ldots, X_{i_n}), & \text{if } i_1, \ldots, i_n \leq k \\ \kappa_{n;\mu_2}(X_{i_1-k}, \ldots, X_{i_n-k}), & \text{if } k < i_1, \ldots, i_n \leq k+\ell \\ 0, & \text{otherwise.} \end{cases} \tag{8.1}$$

For the proof of the fact that $\widetilde{\mu} := \mu_1 \star \mu_2$ satisfies the conditions (8.1), we refer to Lecture 11 of [15]. The uniqueness of a distribution which satisfies (8.1) follows directly from Lemma 8.1(1). Let us also note here that when the formulas connecting moments to free cumulants are used in conjunction to Equation (8.1), it follows in particular that

$$\mu_1 \star \mu_2 \mid \mathbb{C}\langle X_1, \ldots, X_k \rangle = \mu_1, \tag{8.2}$$

and also that $\mu_1 \star \mu_2 \mid \mathbb{C}\langle X_{k+1}, \ldots, X_{k+\ell} \rangle$ is obtained from μ_2 via the relabeling $X_i \mapsto X_{k+i}$, $1 \leq i \leq \ell$.

The versions of free product operations used for c-free and for infinitesimal free independence can be identified by using the same blueprint as above. More precisely, we can proceed as follows.

Proposition and Definition 8.2. *Let k, ℓ be positive integers.*

(1) Let $\mu_1, \nu_1 \in \mathcal{D}_{\text{alg}}(k)$ and $\mu_2, \nu_2 \in \mathcal{D}_{\text{alg}}(\ell)$. We denote $\widetilde{\mu} := \mu_1 \star \mu_2 \in \mathcal{D}_{\text{alg}}(k+\ell)$. There exists a distribution $\widetilde{\nu} \in \mathcal{D}_{\text{alg}}(k+\ell)$, uniquely determined, such that the c-free cumulants of $(\widetilde{\mu}, \widetilde{\nu})$ fulfill the following condition: for every $n \in \mathbb{N}$ and $i_1, \ldots, i_n \in \{1, \ldots, k+\ell\}$, one has

$$\kappa^{(c)}_{n;(\widetilde{\mu},\widetilde{\nu})}(X_{i_1}, \ldots, X_{i_n}) = \begin{cases} \kappa^{(c)}_{n;(\mu_1,\nu_1)}(X_{i_1}, \ldots, X_{i_n}), & \textit{if } i_1, \ldots, i_n \leq k \\ \kappa^{(c)}_{n;(\mu_2,\nu_2)}(X_{i_1-k}, \ldots, X_{i_n-k}), & \textit{if } k < i_1, \ldots, i_n \leq k+\ell \\ 0, & \textit{otherwise.} \end{cases} \tag{8.3}$$

The couple $(\widetilde{\mu}, \widetilde{\nu})$ is called the c-free product *of the couples (μ_1, ν_1) and (μ_2, ν_2); we will use for it the notation*

$$(\widetilde{\mu}, \widetilde{\nu}) = (\mu_1, \nu_1) \star_c (\mu_2, \nu_2). \tag{8.4}$$

(2) Let $\mu_1 \in \mathcal{D}_{\text{alg}}(k)$, $\mu_1' \in \mathcal{D}'_{\text{alg}}(k)$ and $\mu_2 \in \mathcal{D}_{\text{alg}}(\ell)$, $\mu_2' \in \mathcal{D}'_{\text{alg}}(\ell)$. We denote $\widetilde{\mu} := \mu_1 \star \mu_2 \in \mathcal{D}_{\text{alg}}(k+\ell)$. There exists a distribution $\widetilde{\mu}' \in \mathcal{D}'_{\text{alg}}(k+\ell)$, uniquely determined, such that the infinitesimal free cumulants of $(\widetilde{\mu}, \widetilde{\mu}')$ fulfill the following condition: for every $n \in \mathbb{N}$ and $i_1, \ldots, i_n \in \{1, \ldots, k+\ell\}$, one has

$$\kappa'_{n;(\widetilde{\mu},\widetilde{\mu}')}(X_{i_1},\ldots,X_{i_n}) = \begin{cases} \kappa'_{n;(\mu_1,\mu'_1)}(X_{i_1},\ldots,X_{i_n}), & \text{if } i_1,\ldots,i_n \le k \\ \kappa'_{n;(\mu_2,\mu'_2)}(X_{i_1-k},\ldots,X_{i_n-k}), & \text{if } k < i_1,\ldots,i_n \le k+\ell \\ 0, & \text{otherwise.} \end{cases} \tag{8.5}$$

The couple $(\widetilde{\mu},\widetilde{\mu}')$ *is called the* infinitesimal free product *of the couples* (μ_1,μ'_1) *and* (μ_2,μ'_2)*; we will use for it the notation*

$$(\widetilde{\mu},\widetilde{\mu}') = (\mu_1,\mu'_1) \star_B (\mu_2,\mu'_2). \tag{8.6}$$

Proof. (1) The existence of $\widetilde{\nu}$ is a special case of the general result about free product of C-ncps spaces proved in [7]. Uniqueness follows from Lemma 8.1(2).

(2) The existence of $\widetilde{\mu}'$ is a special case of the general construction of a free product of I-ncps spaces which is e.g. discussed in Section 2 of [11]. Uniqueness follows from Lemma 8.1(3). □

Remark 8.3. (1) Analogously to what we had for $\mu_1 \star \mu_2$ in Equation (8.2), the cumulant relations stated in Proposition 8.2(1) imply in particular that $\widetilde{\nu} \mid \mathbb{C}\langle X_1,\ldots,X_k\rangle = \nu_1$, and those from Proposition 8.2(2) imply that $\widetilde{\mu}' \mid \mathbb{C}\langle X_1,\ldots,X_k\rangle = \mu'_1$. Similar relations (modulo relabeling of $X_1,\ldots,X_\ell$ as $X_{k+1},\ldots,X_{k+\ell}$) hold in connection to the restrictions of $\widetilde{\nu}$ and $\widetilde{\mu}'$ to $\mathbb{C}\langle X_{k+1},\ldots,X_{k+\ell}\rangle$.

(2) In Proposition 8.2 it was sufficient to prescribe how the cumulant functionals $\kappa^{(c)}_{n;(\widetilde{\mu},\widetilde{\nu})}$ and $\kappa'_{n;(\widetilde{\mu},\widetilde{\mu}')}$ act on tuples $(X_{i_1},\ldots,X_{i_n})$. The action of these functionals on general tuples $(P_1,\ldots,P_n) \in \Big(\mathbb{C}\langle X_1,\ldots,X_k\rangle\Big)^n$ could also be explicitly described, if needed, by reducing (via multilinearity) to the case when $P_1,\ldots,P_n$ are monomials, and then by invoking the suitable formulas for cumulants with products as entries. In order to get the latter formulas for functionals $\kappa^{(c)}_{n;(\widetilde{\mu},\widetilde{\nu})}$ one can combine Theorem 2.6 of [8] with Proposition 5.4 of the present paper, while for functionals $\kappa'_{n;(\widetilde{\mu},\widetilde{\mu}')}$ one can use Propositions 3.15 and 4.3 from [11].

8.4. Proof of Theorem 1.3. Recall that we have the following data: k,ℓ are positive integers and we are given distributions $\mu_1,\nu_1 \in \mathcal{D}_{\text{alg}}(k)$, $\mu_2,\nu_2 \in \mathcal{D}_{\text{alg}}(\ell)$, such that μ_1,μ_2 are tracial. We consider the free products

$$(\mu_1,\nu_1) \star_c (\mu_2,\nu_2) = (\widetilde{\mu},\widetilde{\nu}) \in \mathcal{D}_{\text{alg}}(k+\ell) \times \mathcal{D}_{\text{alg}}(k+\ell), \text{ and} \tag{8.7}$$

$$(\mu_1,\Psi_k(\nu_1)) \star_B (\mu_2,\Psi_\ell(\nu_2)) = (\widetilde{\mu},\widetilde{\mu}') \in \mathcal{D}_{\text{alg}}(k+\ell) \times \mathcal{D}'_{\text{alg}}(k+\ell), \tag{8.8}$$

where the common $\widetilde{\mu}$ appearing in (8.7) and (8.8) is the free product $\mu_1 \star \mu_2$. We have to prove that $\Psi_{k+\ell}(\widetilde{\nu}) = \widetilde{\mu}'$. In view of Lemma 8.1(3), it will suffice to verify that the equality

$$\kappa'_{n;(\widetilde{\mu},\Psi_{k+\ell}(\widetilde{\nu}))}(X_{i_1},\ldots,X_{i_n}) = \kappa'_{n;(\widetilde{\mu},\widetilde{\mu}')}(X_{i_1},\ldots,X_{i_n}) \tag{8.9}$$

holds for all $n \in \mathbb{N}$ and $1 \le i_1, \ldots, i_n \le k + \ell$.

For the verification of (8.9) we distinguish three cases:

Case 1. All of $i_1, \ldots, i_n$ are in $\{1, \ldots, k\}$.

Case 2. All of $i_1, \ldots, i_n$ are in $\{k+1, \ldots, k+\ell\}$.

Case 3. We are not in Case 1 or Case 2.

In Case 1, the needed verification goes as follows:

$$\begin{aligned}
&\kappa'_{n;(\widetilde{\mu}, \Psi_{k+\ell}(\widetilde{\nu}))}(X_{i_1}, \ldots, X_{i_n}) \\
&= \sum_{m=1}^{n} \kappa^{(c)}_{n+1;(\widetilde{\mu},\widetilde{\nu})}(X_{i_m}, \ldots, X_{i_n}, X_{i_1}, \ldots, X_{i_m}) \quad \text{(by Theorem 1.4)} \\
&= \sum_{m=1}^{n} \kappa^{(c)}_{n+1;(\mu_1,\nu_1)}(X_{i_m}, \ldots, X_{i_n}, X_{i_1}, \ldots, X_{i_m}) \quad \text{(by Proposition 8.2(1))} \\
&= \kappa'_{n;(\mu_1, \Psi_k(\nu_1))}(X_{i_1}, \ldots, X_{i_n}) \quad \text{(by Theorem 1.4)} \\
&= \kappa'_{n;(\widetilde{\mu}, \widetilde{\mu}')}(X_{i_1}, \ldots, X_{i_n}),
\end{aligned}$$

where at the latter equality we invoke Proposition 8.2(2) in connection to how $(\widetilde{\mu}, \widetilde{\mu}')$ is defined in (8.8).

The verifications of Cases 2 and 3 are analogous, where in Case 2 we must at some point shift the indices to $i_1 - k, \ldots, i_n - k \in \{1, \ldots, \ell\}$, while in Case 3 both sides of the required equality (8.9) come out as equal to 0. $\square$

We note that, when expressing the content of Theorem 1.3 directly in terms of the notions of c-free and infinitesimally free independence, one gets the following statement.

Corollary 8.5. *Let k, ℓ be positive integers, consider the algebra of polynomials $\mathcal{A} = \mathbb{C}\langle X_1, \ldots, X_{k+\ell}\rangle$ and its subalgebras*

$$\mathcal{A}_1 = \mathbb{C}\langle X_1, \ldots, X_k\rangle, \quad \mathcal{A}_2 = \mathbb{C}\langle X_{k+1}, \ldots, X_{k+\ell}\rangle.$$

Let $\widetilde{\mu}, \widetilde{\nu}$ be in $\mathcal{D}_{\text{alg}}(k+\ell)$ and let $\widetilde{\mu}' = \Psi_{k+\ell}(\widetilde{\nu}) \in \mathcal{D}'_{\text{alg}}(k+\ell)$. If the subalgebras $\mathcal{A}_1$ and $\mathcal{A}_2$ are c-freely independent in the C-ncps $(\mathcal{A}, \widetilde{\mu}, \widetilde{\nu})$, then it follows that they are infinitesimally free independent in the I-ncps $(\mathcal{A}, \widetilde{\mu}, \widetilde{\mu}')$.

Proof. Let $\mu_1, \nu_1 \in \mathcal{D}_{\text{alg}}(k)$ be the restrictions of $\widetilde{\mu}, \widetilde{\nu}$ to $\mathcal{A}_1$, and let $\mu_2, \nu_2 \in \mathcal{D}_{\text{alg}}(\ell)$ be obtained from the restrictions of $\widetilde{\mu}, \widetilde{\nu}$ to $\mathcal{A}_2$ by the natural shift of indices (for instance $\mu_2(X_{i_1} \cdots X_{i_n}) := \widetilde{\mu}(X_{i_1+k} \cdots X_{i_n+k})$ for every $n \in \mathbb{N}$ and $1 \le i_1, \ldots, i_n \le \ell$). The hypothesis that $\mathcal{A}_1$ and $\mathcal{A}_2$ are c-freely independent in the C-ncps $(\mathcal{A}, \widetilde{\mu}, \widetilde{\nu})$ amounts to the fact that $(\widetilde{\mu}, \widetilde{\nu}) = (\mu_1, \nu_1) \star_c (\mu_2, \nu_2)$. But then Theorem 1.3 tells us that $(\mu_1, \Psi_k(\nu_1)) \star_B (\mu_2, \Psi_\ell(\nu_2)) = (\widetilde{\mu}, \widetilde{\mu}')$, and the latter equation converts into the conclusion about the infinitesimal free independence of $\mathcal{A}_1$ and $\mathcal{A}_2$ in the I-ncps $(\mathcal{A}, \widetilde{\mu}, \widetilde{\mu}')$. $\square$

8.2. Operations of free additive convolution and the proof of Theorem 1.2

For every $k \in \mathbb{N}$ one has an operation $\boxplus$ of free additive convolution on $\mathcal{D}_{\text{alg}}(k)$, which reflects the operation of adding two freely independent k-tuples of elements in a noncommutative probability space. When going to free cumulants, the definition of $\boxplus$ takes the form of a "linearization" property: for $\mu_1, \mu_2 \in \mathcal{D}_{\text{alg}}(k)$, one has

$$\kappa_{n;\mu_1 \boxplus \mu_2} = \kappa_{n;\mu_1} + \kappa_{n;\mu_2} \tag{8.10}$$

(equality of multilinear functionals on $\mathbb{C}\langle X_1, \ldots, X_k\rangle^n$), holding for every $n \in \mathbb{N}$ – see e.g. Lecture 16 in [15]. In a context, such as the one of the present paper, where cumulants are the primary object of the discussion, the linearization property from Equation (8.10) can in fact be used in order to define the operation $\boxplus$. The extensions of $\boxplus$ to c-free and to infinitesimal free probability can be approached in the same way, as indicated next.

Proposition and Definition 8.6. *Let k be a positive integer.*

(1) Let $\mu_1, \mu_2, \nu_1, \nu_2$ be in $\mathcal{D}_{\text{alg}}(k)$. We denote $\mu_1 \boxplus \mu_2 =: \mu \in \mathcal{D}_{\text{alg}}(k)$. There exists a distribution $\nu \in \mathcal{D}_{\text{alg}}(k)$, uniquely determined, such that for every $n \in \mathbb{N}$ and $i_1, \ldots, i_n \in \{1, \ldots, k\}$ one has

$$\kappa^{(c)}_{n;(\mu,\nu)}(X_{i_1}, \ldots, X_{i_n}) = \kappa^{(c)}_{n;(\mu_1,\nu_1)}(X_{i_1}, \ldots, X_{i_n}) + \kappa^{(c)}_{n;(\mu_2,\nu_2)}(X_{i_1}, \ldots, X_{i_n}). \tag{8.11}$$

The couple (μ, ν) is called the c-free additive convolution *of the couples (μ_1, ν_1) and (μ_2, ν_2); we will use for it the notation*

$$(\mu, \nu) = (\mu_1, \nu_1) \boxplus_c (\mu_2, \nu_2). \tag{8.12}$$

(2) Let μ_1, μ_2 be in $\mathcal{D}_{\text{alg}}(k)$ and let $\mu_1' \mu_2'$ be in $\mathcal{D}'_{\text{alg}}(k)$. We denote $\mu_1 \boxplus \mu_2 =: \mu \in \mathcal{D}_{\text{alg}}(k)$. There exists a distribution $\mu' \in \mathcal{D}'_{\text{alg}}(k)$, uniquely determined, such that for every $n \in \mathbb{N}$ and $i_1, \ldots, i_n \in \{1, \ldots, k\}$ one has

$$\kappa'_{n;(\mu,\mu')}(X_{i_1}, \ldots, X_{i_n}) = \kappa'_{n;(\mu_1,\mu_1')}(X_{i_1}, \ldots, X_{i_n}) + \kappa'_{n;(\mu_2,\mu_2')}(X_{i_1}, \ldots, X_{i_n}). \tag{8.13}$$

The couple (μ, μ') is called the infinitesimally free additive convolution *of the couples (μ_1, μ_1') and (μ_2, μ_2'); we will use for it the notation*

$$(\mu, \mu') = (\mu_1, \mu_1') \boxplus_B (\mu_2, \mu_2'). \tag{8.14}$$

Proof. The existence statement in both (1) and (2) of this proposition follow, by a standard argument, from the corresponding parts of Proposition 8.2. (For instance in part (1), one considers the free product $(\widetilde{\mu}, \widetilde{\nu}) = (\mu_1, \nu_1) \star_c (\mu_2, \nu_2) \in \mathcal{D}_{\text{alg}}(2k) \times \mathcal{D}_{\text{alg}}(2k)$, and defines ν via the prescription that $\nu(X_{i_1} \cdots X_{i_n}) = \widetilde{\nu}\big((X_{i_1} + X_{i_1+k}) \cdots (X_{i_n} + X_{i_n+k}) \big)$, for all $n \in \mathbb{N}$ and $1 \leq i_1, \ldots, i_n \leq k$.) The uniqueness statements concerning the required ν and μ' follow from parts (2) and respectively (3) of Lemma 8.1. $\square$

8.7. Proof of Theorem 1.2. Recall that we have the following data: k is a positive integer and $\mu_1, \nu_1, \mu_2, \nu_2$ are in $\mathcal{D}_{\text{alg}}(k)$, with μ_1, μ_2 tracial. We denote

$$(\mu_1, \nu_1) \boxplus_c (\mu_2, \nu_2) = (\mu, \nu) \in \mathcal{D}_{\text{alg}}(k) \times \mathcal{D}_{\text{alg}}(k) \text{ and} \tag{8.15}$$

$$(\mu_1, \Psi_k(\nu_1)) \boxplus_B (\mu_2, \Psi_k(\nu_2)) = (\mu, \mu') \in \mathcal{D}_{\text{alg}}(k) \times \mathcal{D}'_{\text{alg}}(k), \tag{8.16}$$

where the common "μ" appearing in (8.15) and in (8.16) is $\mu = \mu_1 \boxplus \mu_2$. We have to prove that $\mu' = \Psi_k(\nu)$. In view of Lemma 8.1(3), it will suffice to verify that the infinitesimal free cumulants associated to (μ, μ') and to $(\mu, \Psi_k(\nu))$ agree on all tuples of the form $(X_{i_1}, \ldots, X_{i_n})$. In order to do this verification, we can proceed as follows: we start from the equality

$$\kappa'_{n;(\mu,\mu')}(X_{i_1}, \ldots, X_{i_n}) = \kappa'_{n;(\mu_1,\Psi_k(\nu_1))}(X_{i_1}, \ldots, X_{i_n}) + \kappa'_{n;(\mu_2,\Psi_k(\nu_2))}(X_{i_1}, \ldots, X_{i_n}) \tag{8.17}$$

(which holds by Equation (8.16) and the additivity property of $\boxplus_B$), and we expand both cumulants on the right-hand side of (8.17) as sums "$\sum_{m=1}^{n}$" in the way indicated by Theorem 1.4. The sum of the two resulting "$\sum_{m=1}^{n}$" can be further processed by using the fact that the functionals $\kappa^{(c)}_{n+1;(\mu,\nu)}$ and $\kappa^{(c)}_{n+1;(\mu_1,\nu_1)} + \kappa^{(c)}_{n+1;(\mu_2,\nu_2)}$ agree on tuples of the form $(X_{i_m}, \ldots, X_{i_n}, X_{i_1}, \ldots, X_{i_m})$, with $1 \le m \le n$. In this way we arrive to the equality

$$\kappa'_{n;(\mu,\mu')}(X_{i_1}, \ldots, X_{i_n}) = \sum_{m=1}^{n} \kappa^{(c)}_{n+1;(\mu,\nu)}(X_{i_m}, \ldots, X_{i_n}, X_{i_1}, \ldots, X_{i_m});$$

the latter sum is indeed equal to $\kappa'_{n;(\mu,\Psi_k(\nu))}(X_{i_1}, \ldots, X_{i_n})$, by another application of Theorem 1.4. □

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