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Dot Products and Great-Circle Distances

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It is not uncommon that the first topics discussed in freshman-level calculus-based physics courses are vectors and vector products. As the physical applications of vector products (work and torque) do not appear until later in the term, students sometimes feel that they have been taught something of only academic interest. This note illustrates a straightforward, practical application of dot products: that of computing great-circle distances between various locations on the Earth's surface. Anyone who has traveled by air or sea can relate intuitively to such a calculation.

It can be shown that the minimum path length (geodesic) joining two points on a sphere's surface is a segment of a "great circle"—a circle passing through the intersection of the sphere's surface and the plane defined by the center of the sphere and the two points. This can be thought of as the length of an imaginary string held under tension that is stretched from one point to another on the surface of a globe. Incidentally, this is a good way to check your calculations. Usually this is considered an exercise in spherical trigonometry, but the calculation of such distances is fairly straightforward using dot products. Since these calculations require a knowledge of spherical coordinates, which may be beyond the experience of most elementary physics students, we begin with some definitions.

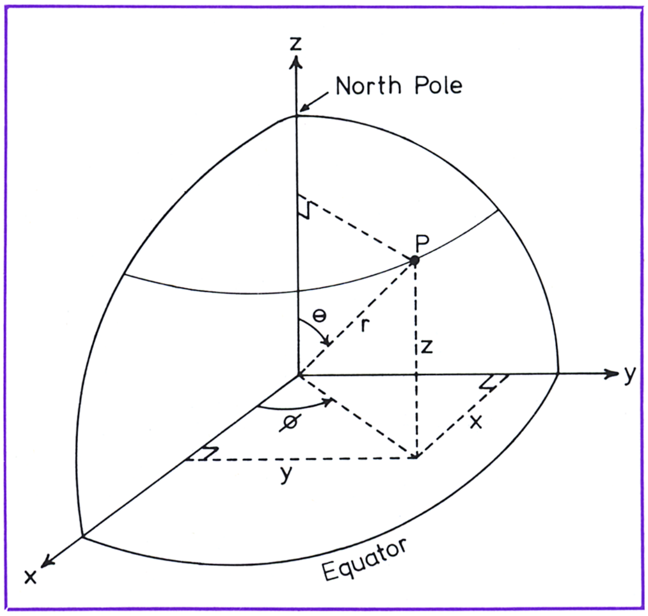

Fig. 1. Defining spherical coordinates. P is any point on the surface of the Earth in the northern hemisphere.

With reference to Fig. 1, consider some point P on the surface of the Earth. The x–y plane is the plane of the equator, and the origin is at the center of the Earth. Point P can be specified by either Cartesian coordinates (x,y,z) or by spherical polar coordinates (r,θ,ϕ) as shown in the diagram. By projecting the position vector r onto the three Cartesian planes, it can be seen that the transformation between Cartesian and spherical coordinates takes the form

$$\vec{r} = (x,y,z) = r(\sin\theta\cos\phi,\ \sin\theta\sin\phi,\ \cos\theta),$$

where r is the radius of the Earth, 6370 km (3960 miles).

Positions on the surface of the Earth are usually specified by latitude and longitude, and it is necessary to know how θ and ϕ relate to these. It is easy to see from Fig. 1 that θ = (90 – latitude) if northern latitudes are considered positive and southern negative. Further, if we take the point at which the x-axis "emerges" through the surface of the Earth in Fig. 1 to be at 0° latitude and 0° longitude, then ϕ is simply the longitude east of Greenwich. In reality this point is somewhere in the Gulf of Guinea.

We are now ready to compute the great-circle distance between two points, A and B, on the Earth's surface (see

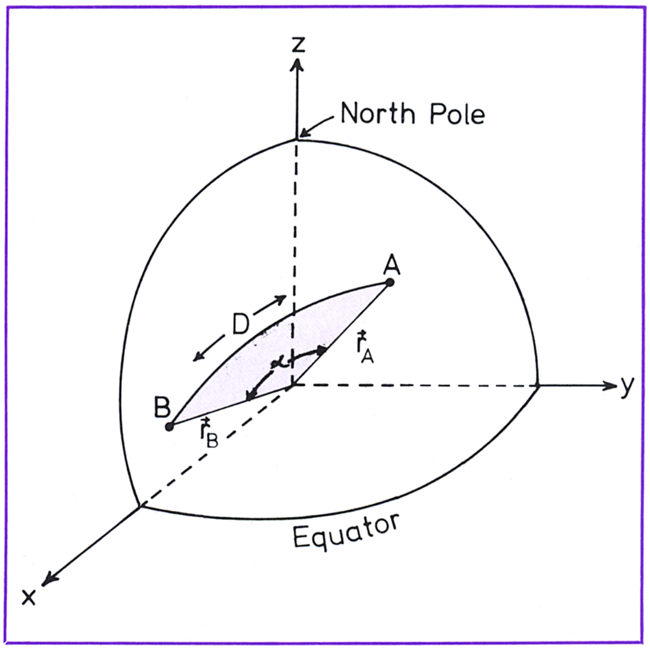

Fig. 2. Illustrating the great-circle distance D between two points A and B on the Earth's surface.

Fig. 2). The angle α subtended by A and B as measured at the center of the Earth is given by

$$\cos\alpha = \vec{r}_A \cdot \vec{r}_B / r^2.$$

(Recall that the dot product of $\vec{r}_A$ and $\vec{r}_B$ is *defined* as $r_A r_B \cos\alpha$). Vectors $\vec{r}_A$ and $\vec{r}_B$ have different components but the same *length* (the radius of the Earth), thus $r_A r_B = r^2$. The great-circle distance D between A and B is then given by $D = r\alpha$ with α measured in radians.

For example, consider the cities of New York and Los Angeles, whose positions in degrees are (40.8 N, 74.0 W) and (34.0 N, 118.3 W), respectively. We then have $(\theta,\phi)_{NY} = (49.2, -74.0)$ and $(\theta,\phi)_{LA} = (56.0, -118.3)$. Now converting to Cartesian coordinates,

$$\vec{r}_{NY} = r(0.209, -0.728, 0.653)$$

and

$$\vec{r}_{LA} = r(-0.393, -0.730, 0.559).$$

Therefore $\vec{r}_{NY} \cdot \vec{r}_{LA} = r^2\,[(0.209)(-0.393) + (-0.728)(-0.730) + (0.653)(0.559)] = 0.814r^2$ and $\alpha = \cos^{-1}(0.814) = 0.619$ rad. Thus D = 3944 km (2452 miles).

The differences between great-circle distances and those computed from drawing a straight line on a plane map can be appreciable. Because the scale of a plane map is a function of latitude (look at any Mercator-projection map of the world), the calculation of the "straight-line" distance is quite involved. We can devise a simple example that will demonstrate the effect.

Suppose that you want to fly from (40 N, 60 W) to (40 N, 120 W). The "straight-line" path would have you fly along the 40° north latitude line all the way; you would be traveling along the arc of a circle whose center is on the Earth's axis somewhere under the north pole. This is illustrated in Fig. 3, where the radius R of your circular trajectory is clearly $R = r \sin\theta$. At 40 N, we have $\theta = 50°$, so $R = 0.77r = 4880$ km. A longitude of 60° corresponds to traversing one-sixth of the complete circle, so the straight-line distance is $(1/6)(2\pi)(4880 \text{ km}) = 5110$ km. The great-circle distance between these two points is just over 100 km shorter, 5008 km. Only if you are traveling between two points having the same longitude or that are on the equator will the straight-line and great-circle distances be the same.

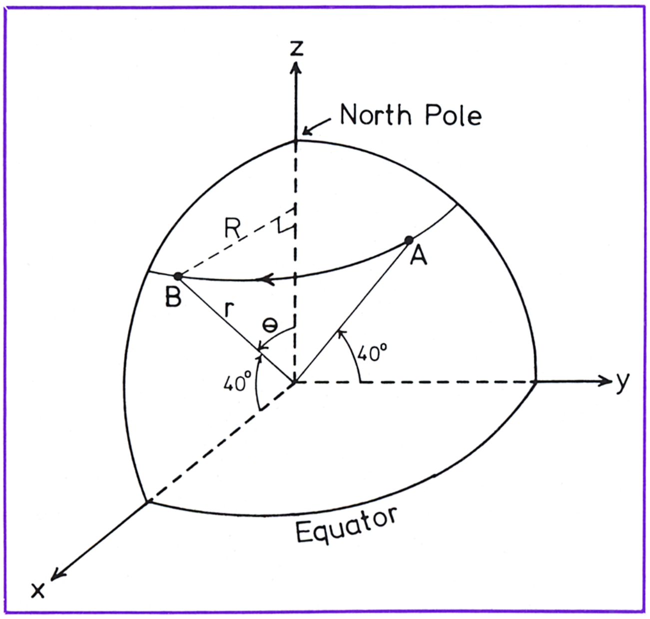

Fig. 3. Illustrating the straight-line distance between two points A and B on the Earth's surface. Both are at 40° north latitude.

What are the distances between your favorite places? Discussion of such problems can be used to motivate qualitative descriptions of non-Euclidean geometry and its relation to relativity theory.

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