Making Sense of Holes In Spaces Using Algebra

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Abstract

Algebraic topology provides a nice method for detecting holes in topological spaces by the use of algebra. It turns out that algebra and holes are related to each other by groups: the fundamental group and homology group. On the matter of the fundamental group, there is a generalization to higher dimensions, called higher homotopy groups. In practice these are more difficult to compute, leading the discussion to go in the direction of homology groups, which are easier to compute. In doing this, we address the two varieties of homology groups, called simplicial and singular homology groups. Even though homology groups are easier to compute, we have to work hard to construct them. To remedy this, we turn to the Eilenberg-Steenrod approach which takes the properties of homology as axioms.

1 Introduction: The Historical Motivation

In this section we will be primarily looking at the development of homology, but we will briefly mention homotopy when it is appropriate. This will start with Riemann in looking at closed curves on a surface. Then we will see the way that Poincaré built upon Riemann's work, which was done by taking an algebraic approach. This algebraic approach looks somewhat like homology in a modern sense, but it is not until we begin to discuss the results of Noether and Lefschetz where we start to get a more complete algebraic picture of homology. The last contribution to the study of homology, which is from Eilenberg and Steenrod, provides us with an axiomatic method for studying homology. By this point it may not be clear as to why we are interested in the study of algebraic topology, so, after going through the history, we will have to talk about the classification problem of topological spaces. This problem will give us an appreciation for the historical background on the matter and will provide motivation for what we will be discussing. I do not want to spend a great deal of time going over the history of algebraic topology, but this will give a pretty good idea of how the interest in the subject progressed.

To begin our account of algebraic topology, we turn to Riemann, who started by considering a system of closed curves C on a surface S. When taking the contour integral $\int_C Xdx + Ydy$, Riemann remarked that the integral vanished if the system of closed curves formed the complete boundary of the region in S [1]. This lead to the discussion of connectivity numbers, which in particular, had importance for when Riemann looked at what would happen if cuts were made in the surface S. As it turns out, the number of cuts needed to make S simply connected is the connectivity number, and for a compact Riemann surface, one needs an even number of cuts $2p$ [1]. In a modern sense, p is what we call genus [1], which came from Clebsch when studying surfaces form an algebraic geometric point of view after Riemann [2].

The next bit of developement came from Poincaré by starting with a geometric object which, with combinatorial data, gives a simplicial complex. These simplicial complexes were used to construct a homology group, which is established by the use of linear algebra and boundary relations [3]. Another concept that came from Poincaré's research is homotopy and the fundamental group, where the aim of homotopy is to study properties of topological spaces and continuous maps [3]. The two major results, however, from Poincaré are Betti numbers and a generalization of Euler characteristic to higher dimensional polyhedra [1]. Both of which, we will discuss when looking at simplicial homology.

Noether was the next to push the study of homology in a more algebraic direction by formulating an algebraic take on Poincaré's geometric approach of simplicial homology [3]. Noether realized that homology groups are abelian which, up to this point, homology was just Betti numbers and torsion coefficients (something introduced by Poincaré) [1], this indicated that there was something more algebraic lurking within homology that needed to be taken into consideration. Lefschetz introduced a way of extending simplicial homology to singular homology. This was done by taking continuous maps from standard simplices to any topological space and using algebraic properties of singular chain complexes [3]. The last major contribution I want to bring to attention is the work done by Eilenberg and Steenrod. The two considered a small number of properties as axioms to characterize a theory of homology. The result of this approach being, that it added an elegance and coherentness to homology, and provided quick techniques for computing homology groups [3].

If we take all of what has been established at the surface level, it seems as though mathematicians were developing an algebraic way of studying surfaces, which is not entirely false. However, each piece of the history has a strinkingly subtle result, essentially, there is some sort of algebraic property of the space. For example, the homology group of a topological space is a property of that space, which may lead one to ask if there is some sort of relation between spaces with the same homology group. What follows from this question is known as the classification problem of topological spaces, which is the investigation into whether topological spaces are homeomorphic or not [3].

To solve this problem, we have to either show that two topological spaces are homeomorphic or that there is no such homeomorphism. Looking to show that no such homeomorphisms exist, the solution shifted to algebra by associating invariant objects in the sense that homeomorphic spaces have the same algebraic object [3]. These algebraic objects may be the Betti numbers, as previously mentioned, or they could be groups associated with the space, in either case it gives us a better way of treating the problem by considering how spaces may share algebraic structures [3]. In the case of homotopy, we look at the classification of equivalence relations, which lead to the notion of the fundamental group [3]. So the goal in studying algebraic topology is to look for algebraic properties of spaces to determine if two given spaces are homeomorphic.

2 Category Theory

One useful tool that will be necessary in our study of algebraic topology is category theory. In this section we will spend time discussing some of the fundamentals of category theory, but note that no use of these concepts will occur until we begin to discuss homology. However, it is something I want to establish earlier rather than later, so as not to disrupt the pacing of the discussion. With this in mind, I will draw attention to some particular categories and functors which will be of importance later. It should be said that category theory is a very deep rabit hole, and for the level in which we are treating algebraic topology, we only need to poke a finger in, so this section will focus on defining terminology.

2.1 Categories and Functors

The motivation for categories can be best put as noticing that there are many instances in different contexts where things behave in a similar manner. For example products, in general, are defined similarly for different contexts, so it makes sense to generalize the idea of a product insofar as to make the product act on given objects [4]. It is not just products that enjoy this generalization, homomorphisms look similar whether it's between groups, rings, fields or any other variety of algebraic structure. This leads to our first definition.

Definition 2.1.1. *A category C consists of objects $ob(C)$ and morphisms $Hom_C(X,Y)$ associated with any two objects $X, Y \in ob(C)$, where X is the domain and Y is the codomain, in which we have that morphisms are subjected to the following conditions:*

i. We can compose two morphisms if the codomain of one morphism is the domain of the other morphism;

ii. For any object $X \in ob(C)$ there is an identity morphism $id_X \in Hom_C(X,X)$ such that for all $f : X \to A$ and $g : B \to X$ we have $f \circ id_X = f$ and $id_X \circ g = g$,

The basic notion of a category should feel familiar. If we consider the category *Sets* whose objects are sets and morphisms are maps between the sets [5] or the category *Groups* where the objects are groups and the morphisms are group homomorphisms [5] . Then it becomes obvious that a category generalizes the concepts we are already aquainted with. For instance, although completely different in nature, the maps between sets and groups appear to be doing the same thing, i.e., mapping elements from one set (group) to the other. Like invertible maps in the ordinary sense, in the context of categories, we find something similar.

Definition 2.1.2. *Let C be a category and $X, Y \in ob(C)$ with morphisms $f : X \to Y \in Hom_C(X,Y)$. The morphism $f : X \to Y$ is said to be an isomorphism if there exists a morphism $g : Y \to X$, called the inverse of f, such that $g \circ f = id_X$ and $f \circ g = id_Y$.*

The inverse morphism is unique, which we can show by considering, $g_1, g_2 : Y \to X$ and $f : X \to Y$ an isomorphism, then,

$$\begin{aligned} g_2 = g_2 \circ id_Y = g_2 \circ (f \circ g_1) &= (g_2 \circ f) \circ g_1 \\ &= id_X \circ g_1 = g_1 \end{aligned}$$

The next part of our discussion directs us toward looking at the way in which we relate one category to another by means of functors [4]. It does not have an intuitive analogy like we saw before, where we had the idea of objects being sets and morphisms being maps between sets, but it is still necessary to understand if we want to be able to use category theory. There are two kinds of functors, one being covariant and the other being contravariant, but by defining one of them, we can get the other one for free.

Definition 2.1.3. *A covariant functor $F : C \to D$ from a category C to a category D assigns to each*

$X \in ob(C)$ an object $F(X) \in ob(D)$ and to every morphism $f \in Hom_C(X,Y)$ with $X, Y \in ob(C)$ a morphism $F(f) \in Hom_D(f(X), f(Y))$ such that:

i. $F(g \circ f) = F(g) \circ F(f)$ $\forall f \in Hom_C(X,Y)$ *and* $\forall g \in Hom_C(Y,Z)$;

ii. $F(id_X) = id_{F(X)}$ $\forall X \in ob(C)$

Remark. *For contravariant functors we note that all that needs to change within this definition is:*

$F(f) \in Hom_D(F(Y), F(X))$ *and* $F(g \circ f) = F(f) \circ F(g)$.

It is a good time to introduce a tool that we can use to look at morphisms between objects in a category. This being the concept of a diagram which is something that is indispensable for our purposes.

Definition 2.1.4. *A diagram in a category C is a directed multigraph whose vertices are objects in C and whose arrows are morphisms in C.*

We need not concern ourselves with what it means for something to be a directed multigraph, we only need to understand that the diagram shows how a mapping between objects under morphisms occur. To see an example of a diagram, consider the following square diagram where $A, A', B, B' \in ob(C)$ with morphisms ∂, $\partial^{'}$, ϕ and ϕ'.

$$\begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "\partial'"] & B \arrow[d, "\partial"] \\ A' \arrow[r, "\phi'"] & B' \end{tikzcd}$$

Figure 2.1: A Square Commutative Diagram

If we have a diagram, we may want to know whether it commutes or not. The reason for this being that commutation will tell us if the composition of morphisms takes us from one place to the other in the same way. This idea will be explored when we actually have something to work with, like when looking at homology, but for now we just need to understand it in a fundamental way.

If we look at figure 2.1, this diagram commutes if $\partial \circ \phi = \phi^{'} \circ \partial^{'}$ because either side takes A to B'. To confirm this, notice that ϕ takes A to B and ∂ takes B to B', by following the composition. Doing this for the other composition gives us, ∂' takes A to A' and ϕ' takes A' to B'. In other words, it does not matter if we take ϕ first or $\partial^{'}$ first, as long as we follow which ever one with the appropriate morphism. To summarize, this sort of relationship between objects and morphisms means that we do not need to look at any specific detail about the individual morphism.

This part of the section is meant to bring to attention some of the very basics of category theory that will be used in our discussion on algebraic topology. It cannot be over-emphasized that this is barely scratching the surface for what there is, but this is all that we will want to worry about.

2.2 Basic Categories

After having introduced the elementary ideas of category theory, I would like to bring some of the basic categories to light. In particular, I will define categories that we will find useful in the study of algebraic topology. To start, we will look at categories that pertain to topology, and then we will define categories that deal with algebraic structures.

Definition 2.2.1. *The category $\mathcal{T}op$ is the category of topological spaces, which consists of topological spaces as objects and continuous maps as morphisms.*

Remark. *Isomorphisms in this category are homeomorphisms.*

Definition 2.2.2. *The category $\mathcal{T}op^2$ is the category of topological pairs, which consists of pairs of a topological space and a subspace, as objects, and continuous maps that restrict to a map of the subspaces.*

Remark. *If the morphism is an isomorphism, then the isomorphisms is of this category are homeomorphisms that restrict to a homeomorphism of the subspaces.*

These are the two important categories (in regard to topology) for our discussion, now we will look at the categories which deal with groups.

Definition 2.2.3. *The category Group is the category of groups, which has groups as objects and group homomorphisms as morphisms.*

Remark. *If the morphism is an isomorphism, then the isomorphisms are group isomorphisms.*

Definition 2.2.4. *The category Ab is the category of abelian groups, which has abelian groups as objects, and group homomorphisms as morphisms.*

Remark. *The same thing we said about isomorphisms in the category Group holds for the category Ab.*

If we happen to encounter any new categories, we will define them when necessary. This section is only meant to expose the reader to some of the basics.

3 Homotopy and The Fundamental Group

The first aspect of algebraic topology I want to discuss is that of the fundamental group. The fundamental group will give us a taste for the type of subject matter we will be dealing with. The fundamental group, put briefly, will detect holes in a topological space [3]. This is done by looking at homotopic equivalent loops on the space, which we will see forms a group. But before we get into doing this, we will work our way through some of the basics of topology to get oriented.

3.1 Topological Spaces, Countinous Maps, and Paths/Loops

We will begin by looking at topological spaces, which will serve as the foundation for continuous maps. This type of map will be necessary when looking at paths and loops, but it will be useful when discussing homology as well. We more or less want to get a feel for what a topological space is, so when talking about them there is no confusion. As for continuous maps, we need to know the nature of these maps in order to go from one topological space to another topological space, or even look at homeomorphisms. After we have done this, I would like to take a look at paths and loops in topological spaces, which we will build on in the next subsection.

The first way we can look at a topological space is geometrically. In this sense we can think of topology as a sort of distanceless geometry, one in which we need to relax the definition when a solid is subjected to a deformation. This gives us an understanding of a topological space as having the properties of geometric figures that remain valid after the figures are subjected to deformations [7]. On the other hand, topology can be looked at in a more abstract way. This brings us to the definition of a topological space.

Definition 3.1.1. *Let X be a nonempty set and $\mathcal{T}$ be a collection of subsets of X. Then $\mathcal{T}$ is called a topology on X and the subsets in $\mathcal{T}$ are called open sets of X if the following are satisfied:*

i. $\emptyset$ and X belong to $\mathcal{T}$;

ii. The union of any number of open sets is an open set ($\in \mathcal{T}$);

iii. The intersection of a finite number of open sets is an open set ($\in \mathcal{T}$).

At first glance, this definition may feel vague. This vagueness is because mathematicians were looking for a definition that was as broad as possible, but also narrow enough so that standard theorems about familiar spaces would hold in topological spaces, generally [8]. One difference we can see between the concept of an open set in real analysis and topology is that in the former, we tend to rely on a $\epsilon - \delta$ formulation, while in the latter, an open set is any set that belongs to the topology. With this way of looking at open sets, when dealing with the continuity of functions, we will be looking for functions that preserve the openess of a set.

Definition 3.1.2. *Let $(X, \mathcal{T})$ and $(Y, \mathcal{J})$ be two topological spaces. A function $f : X \to Y$ is said to be continuous if $f^{-1}(U) \subset X$ is an open set in X for every open set $U \in Y$. If there is a continuous inverse $g : Y \to X$ of f, then f is said to be a homeomorphism.*

Remark. *The inverse image f^{-1} does not mean that the function is invertible. We are just considering the set $f^{-1}(U)$ contained in X.*

For the most part, homeomorphic spaces can be thought of as being the same, and as previously discussed, this is something we are going to be interested in looking at. As it stands, our skirmish with topology has concluded since the idea of paths will lead us astray. There are some other lingering concepts that I have opted out of talking about for the time being, but these will be dealt with when their relevance is apparent. I will finish this subsection by introducing paths and loops in topological spaces.

Definition 3.1.3. *Let $(X, \mathcal{T})$ be a topological space and $f : [0,1] \to X$ a continuous map. If $f(0) = a$ and $f(1) = b$, then f is said to be a path connecting the points $a, b \in X$.*

Remark. *Sometimes we will use I to denote* $[0,1]$.

This idea is pretty intuitive, and we will see when looking at homotopy that the idea of paths becomes more interesting. What we will be more interested in, however, is the concept of a loop. To get the definition of a loop from the definition of a path, all we have to do is modify a path so that the initial point $f(0)$ is the same as the final point $f(1)$. In other words, a loop is a path that starts and finishes at the same point.

Definition 3.1.4. *Let $(X, \mathcal{T})$ be a topological space and $f : [0,1] \to X$ a continuous map. If $f(0) = a = f(1)$, then f is said to be a loop based at the point $a \in X$.*

We will use loops for our discussion of homotopy, but we can, and sometimes will, use paths instead of loops in the proceeding subsection on homotopy.

3.2 Homotopic Paths/Loops

Now that we have an idea of what paths and loops are, we want to consider what happen if there happens to be more than one path in a space that share the same initial and final point in a space. If we are looking at two loops, for example, then one could ask "how are these loops related to each other?" The answer to this question is the essence of homotopy, but for now, we define a homotopy in a general sense.

Definition 3.2.1. *Let X and Y be topological spaces and $f, g : X \to Y$ be continuous maps between the two spaces. Then we say that f is homotopic to g if there is a map $H : X \times I \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for each x. The map H is called a homotopy between f and g.*

The idea of paths is not completely captured here because we can observe that f, g are continuous maps that take X into Y, where recall for paths, we map I (the unit interval) into a topological space. To introduce paths into the picture, and hence loops, we consider the following definition.

Definition 3.2.2. *Let X be a topological space and $f, g : I \to X$ be two continuous paths in X, with $f(0) = a = g(0)$ and $f(1) = b = g(1)$ for $a, b \in X$. Then f is said to be path homotopic to g if there is a map $H : I \times I \to X$ such that $H(s, 0) = f(s)$, $H(s, 1) = g(s)$, $H(0, t) = a$ and $H(1, t) = b$ for all $s \in I$ and for all $t \in I$.*

A common way of looking at what is going on here is to treat t as if it were time, then the homotopy H represents a continuous deforming of the map f into the map g, as t goes from 0 to 1 [8]. We can readily understand that if f, g are loops in the topological space X, then the homotopy between them is very much similar to that of path homotopy. In fact, all we have to change is that $H(0,t) = a = H(1,t)$ for all $t \in I$ in definition 3.2.2.

It turns out that there is something lurking within path homotopy (and loops), one which serves as the foundation for the fundamental group. To get an idea of what this "something" is, suppose we were to consider an equivalence relation defined on homotopic paths. In other words, we suppose $f \simeq g$, meaning f and g are path homotopic. Before we see why this equivalence relation is important, moreover, why this is an equivalence relation to begin with, we need to introduce something known as the Pasting Lemma. Introducing the Pasting Lemma here makes sense, as it will be used more than once throughout this subsection. For a proof of this lemma, we will follow [6].

Lemma 3.2.3. *Let X be a topological space and A, B be closed subsets in X such that $X = A \cup B$. Given a topological space Y, if $f_1 : A \to Y$ and $f_2 : B \to Y$ are continuous maps such that $f_1(x) = f_2(x)$, $\forall x \in A \cap B$, then the map $f : X \to Y$, $x \mapsto \begin{cases} f_1(x), & if\ x \in A \\ f_2(x), & if\ x \in B \end{cases}$ is continuous.*

Proof. The map $f : X \to Y$ is well-defined, so we just need to show that it is continuous. Let K be a closed set in Y. Then $f^{-1}(K) = (A \cup B) \cap f^{-1}(K) = (A \cap f^{-1}(K)) \cup (B \cap f^{-1}(K)) = {f_1}^{-1}(K) \cup {f_2}^{-1}(K)$. Since f_1 and f_2 are continuous, then ${f_1}^{-1}(K)$ and ${f_2}^{-1}(K)$ are both closed in X. This means that for $f^{-1}(K)$ being the union of two closed sets is closed in X. Hence, f is continuous. □

Remark. *A map $f : X \to Y$ is continuous if and only if $f^{-1}(A) \subset X$ is a closed set in X for every closed set A in Y.*

Now that we have the Pasting Lemma available, we will show that the aforementioned equivalence relation is indeed an equivalence relation, I will use [8] as a guide to show this.

Lemma 3.2.4. *The relation $\simeq$ is an equivalence relation.*

Proof. Showing that reflexivity holds is trivial, since for any path, the path is homotopic to itself. As for symmetry, we have to consider the homotopy which undos the homotopy between f, g. Namely, if $H(s,t)$ is a homotopy between f and g, then $H'(s, 1-t)$ is the homotopy between g and f which means that $f \simeq g \Rightarrow g \simeq f$. Finally, for transitivity, if $f \simeq g$, $g \simeq h$, with homotopy $H(s,t)$ and $H'(s,t)$, respectively. Then we can construct a homotopy

$$H''(s,t) = \begin{cases} H(s,t) & : t \in \left[0, \frac{1}{2}\right] \\ H'(s,t) & : t \in \left[\frac{1}{2}, 1\right] \end{cases}$$

which is continuous by the Pasting Lemma and is a homotopy between f and h, hence, $f \simeq h$. □

Remark. *Just to be clear, the equivalence classes here are homotopic paths, which we will denote by* $[f]$.

Since this is an equivalence relation, the next thing we are going to do is try to piece together a group structure by using the homotopic equivalent paths/loops. In fact, it should be said that what we will be most interested in are loops as opposed to paths.

3.3 Groups and Homotopic Loops

To be able to discuss the fundamental group, we will recall the definition of a group and some other ideas that arise from group theory. As a disclaimer, I want it to be known that there will be some aspects of group theory that will not be used until we move into the topic of homolgy. That being said, with a sense of what a group is, we can push toward establishing an operation on the equivalence classes of loops. Once we know that this operation is well-defined, we can begin to look at the group structure of homotopic loops. The technical nature of the fundamental group is in some sense intuitive, but in practice it can be difficult to execute. The fundamental group, as a reminder, can be thought of as means for detecting holes in a space. We won't really begin to see this until the next subsection, but it is something to keep in mind for along the way.

Definition 3.3.1. *Let G be a set with a binary operation $\times$ defined so that for any $a, b \in G$, $a \times b \in G$. Then G is said to be a group if the following are satisfied:*

i. $a \times (b \times c) = (a \times b) \times c \; \forall a, b, c \in G$;

ii. There is an element $e \in G$ such that for any $a \in G$ $e \times a = a \times e = a$;

iii. For any $a \in G$ there is an element $a^{-1} \in G$ such that $a \times a^{-1} = a^{-1} \times a = e$.

Remark. $\times$*, as a reminder, does not mean multiplication.*

Remark. *The group identity is unique and the inverse of an element is unique.*

Remark. *An important example of a group that will pop up in algebraic topology is the trivial group whose only element is the identity itself, which is represented by* 0 *or* $\{0\}$.

It is important to keep in mind is that a group is a set, albeit, with an operation, and we will use this fact in the following way. Since a group has an underlying set, a natural question might be to ask when given a subset H of G, where G is a group, does H also have a group structure? The answer to this question is found in the notion of a subgroup.

Definition 3.3.2. *Let G be a group, and let H be a subset of G. Then H is called a subgroup of G if H is itself a group, under the operation of G.*

From this definition, this would mean that one would have to check the group axioms for the subset H. But, we can note that associativity is inherited by a subgroup [10], so we can take that axiom as a given. Still, satisfying these axioms is cumbersome to check if H is a subgroup, we can consider the following corollary.

Corollary 3.3.3. *Let G be a group and let $H \subseteq G$. Then H is a subgroup of G if and only if $H \neq \emptyset$ and $ab^{-1} \in H$, $\forall a, b \in H$.*

The notion of a subgroup will mostly be important when considering groups that arise in homology. Anyhow, the last big idea from group theory I want to bring to attention is a group homomorphism. A homomorphism (different from homeomorphism) is a map that acts between groups, preserving some structure, namely algebraic infromation [10].

Definition 3.3.4. *Let G and G' be groups, and let $\phi : G \to G'$ be a function. Then ϕ is said to be a group homomorphism if $\phi(ab) = \phi(a)\phi(b)$, $\forall a, b \in G$.*

Remark. *If a group homomorphism is bijective, then it is said to be a group isomorphism, which in this case does preserves structure.*

There is another thing to consider when dealing with homomorphisms, this being the kernel of a group homomorphism. The kernel of a homomorphism is necessary in algebraic topology, and is used in the very definition of the homology group.

Definition 3.3.5. *Let $\phi : G \to G'$ be a group homomorphism. Then the set $\{x \in G : \phi(x) = e\}$ is called the kernel of ϕ, which we denote $ker(\phi)$.*

Remark. *Since the image of ϕ is in G', it should be emphasized that e in this set denotes the identity in G' and not in G.*

Thus far, in this subsection all that we have been working toward is a very general idea of what a group is. It turns out that we only need a general idea of a group in order to look at the group structure of homotopic equivalence classes. But, as we now know, it is necessary that we have an operation that ensures the equivalence classes, under this operation, remain equivalence classes. We also know that once we have such an operation, all that we need to do afterward is check the group axioms. When the dust settles, we should be left with an algebraic structure consisting of homotopy classes.

We begin by defining an operation on loops. This operation induces a well-defined operation on the equivalence classes [8].

Definition 3.3.6. *If f, g are loops in the space X based at a point $x_0 \in X$, then their product $f * g : I \to X$ is defined by*

$$(f * g)(t) = \begin{cases} f(2t) & : \ t \in \left[0, \frac{1}{2}\right] \\ g(2t-1) & : \ t \in \left[\frac{1}{2}, 1\right] \end{cases}.$$

Remark. *This operation is continuous by the pasting lemma.*

This operation also holds for the equivalence classes (as previously noted), this means that for the equivalence clesses, this operation does the following, $[f] * [g] = [f * g]$. This only covers one part of showing that this is a group operation, the next thing to be shown is that associativity holds. The key idea behind this is ensuring that there is a homotopy between loops $f * (g * h)$ and $(f * g) * h$, we will consider the proof as laid out in [3].

Proposition 3.3.7. *If f, g, h are loops based at a point $x_0 \in X$, where X is a space, then $f * (g * h) \simeq (f * g) * h$.*

Proof. To start, we will define these products, so we define $f * (g * h) : I \to X$ and $(f * g) * h : I \to X$ given by,

$$(f * (g * h))(s) = \begin{cases} f(2s) & : \ s \in \left[0, \frac{1}{2}\right] \\ (g * h)(2s-1) & : \ s \in \left[\frac{1}{2}, 1\right] \end{cases}$$

$$= \begin{cases} f(2s) & : \ s \in \left[0, \frac{1}{2}\right] \\ g(4s-2) & : \ s \in \left[\frac{1}{2}, \frac{3}{4}\right] \\ h(4s-3) & : \ s \in \left[\frac{3}{4}, 1\right] \end{cases}$$

and

$$((f * g) * h)(s) = \begin{cases} (f * g)(2s) & : \ s \in \left[0, \frac{1}{2}\right] \\ h(2s-1) & : \ s \in \left[\frac{1}{2}, 1\right] \end{cases}$$

$$= \begin{cases} f(4s) & : \ s \in \left[0, \frac{1}{4}\right] \\ g(4s-1) & : \ s \in \left[\frac{1}{4}, \frac{1}{2}\right] \\ h(2s-1) & : \ s \in \left[\frac{1}{2}, 1\right] \end{cases}$$

respectively. Both of these are well-defined and continuous by the pasting lemma. Now, we define the homotopy,

$$F(s,t) = \begin{cases} f\left(\frac{4s}{(1+t)}\right) & : \ s \in \left[0, \frac{(1+t)}{4}\right] \\ g(4s-1-t) & : \ s \in \left[\frac{(1+t)}{4}, \frac{(2+t)}{4}\right] \\ h\left(1 - \left(\frac{4(1-s)}{(2-t)}\right)\right) & : \ s \in \left[\frac{(2+t)}{4}, 1\right] \end{cases}$$

which is well-defined and continuous by the pasting lemma. We check that this is a homotopy by checking that for $t = 0$ we have one loop, and for $t = 1$ we have the other. So, when $t = 0$ we have,

$$F(s,0) = \begin{cases} f(4s) & : \ s \in \left[0, \frac{1}{4}\right] \\ g(4s-1) & : \ s \in \left[\frac{1}{4}, \frac{1}{2}\right] \\ h(2s-1) & : \ s \in \left[\frac{1}{2}, 1\right] \end{cases}$$

which corresponds to $((f * g) * h)(s)$, and for $t = 1$ we have,

$$F(s,1) = \begin{cases} f(2s) & : \ s \in \left[0, \frac{1}{2}\right] \\ g(4s-2) & : \ s \in \left[\frac{1}{2}, \frac{3}{4}\right] \\ h(4s-3) & : \ s \in \left[\frac{3}{4}, 1\right] \end{cases}$$

which corresponds to $(f*(g*h))(s)$. Thus, these loops are homotopic, and hence, associativity holds. □

The next thing that we will want to show is true is the existance of an identity, we will again turn to [3] for a proof of this.

Proposition 3.3.8. *If f is a loop based at a point x_0 in a space X, and $\delta : I \to X$ such that $\delta(t) = x_0$ for all $t \in I$, then $f * \delta \simeq f$ and $\delta * f \simeq f$.*

Proof. Note that it's sufficient to show that $f * \delta$ because showing $\delta * f$ follows in the same way. So, we start by defining,

$$(f * \delta)(s) = \begin{cases} f(2s) & : \ s \in \left[0, \frac{1}{2}\right] \\ \delta(2s-1) & : \ s \in \left[\frac{1}{2}, 1\right] \end{cases}$$

which since $\delta(s) = x_0$ for all s, we have then,

$$(f * \delta)(s) = \begin{cases} f(2s) & : \ s \in \left[0, \frac{1}{2}\right] \\ x_0 & : \ s \in \left[\frac{1}{2}, 1\right] \end{cases}$$

and hence, this is a loop, now if we define the following homotopy,

$$F(s,t) = \begin{cases} f(\frac{2s}{(1+t)} & : \ s \in \left[0, \frac{(1+t)}{2}\right] \\ x_0 & : \ s \in \left[\frac{(1+t)}{2}, 1\right] \end{cases}$$

then this is a homotopy between $f * \delta$ and f, therefore, $f * \delta \simeq f$. □

In showing that there is an inverse element, we note that the reverse of a loop $f(t)$ is just $f(1-t)$. For one last time, we will turn to [3].

Proposition 3.3.9. *If f is a loop based at a point x_0 in X, then $f * f^{-1} \simeq \delta$ and $f^{-1} * f \simeq \delta$.*

Proof. As was the case in the previous proposition, showing $f * f^{-1}$ is enough because a similar argument applies to $f^{-1} * f$. To begin, define $f * f^{-1} : I \to X$ by,

$$(f * f^{-1})(s) = \begin{cases} f(2t) & : \ s \in \left[0, \frac{1}{2}\right] \\ f^{-1}(2s-1) & : \ s \in \left[\frac{1}{2}, 1\right] \end{cases},$$

because $f^{-1}(t) = f(1-t)$, we have,

$$(f * f^{-1})(s) = \begin{cases} f(2t) & : \ s \in \left[0, \frac{1}{2}\right] \\ f^{-1}(2-2s) & : \ s \in \left[\frac{1}{2}, 1\right] \end{cases},$$

meaning that this is indeed a loop. Lastly, define the homotopy,

$$F(s,t) = \begin{cases} f(2s(1-t)) & : \ s \in \left[0, \frac{1}{2}\right] \\ f((2-2s)(1-t)) & : \ s \in \left[\frac{1}{2}, 1\right] \end{cases},$$

and hence, we have that $f * f^{-1}$ is homotopic to δ. $\square$

Remark. *All of what we have done up to this point holds in the context of homotopy classes of loops.*

The reason that we did this work to establish the validity of these properties is because it makes the definition of the fundamental group more concise than if we did not show the validity of the group axioms. In other words, we are able to refer to these properties in the definition of the fundamental group.

Definition 3.3.10. *The group $\pi_1(X, x_0)$ consisting of homotopy classes of loops under the product defined in definition 3.3.6 is called the fundamental group of the topological space X based at the point x_0.*

In the next subsection, we will get a feel for the fundamental group by seeing how we would compute it in a simple example.

3.4 Computation and Homotopy Groups

With the fundamental group at our disposal, it is time that we make use of it. We will start by discussing the concept of a space being simply connected and see its implication. Then we will consider an example, which will be computing the fundamental group of a circle. Before departing from this aspect of algebraic topology, I want to look beyond the fundamental group, which will bring us to the discussion of homotopy groups.

Definition 3.4.1. *A topological space X is said to be simply connected if is path-connected and if the fundamental group is trivial for every $x_0 \in X$.*

Remark. *The fundamental group being trivial is often denoted by $\pi_1(X, x_0) = 0$.*

To further the idea of simply connected, we will consider how the space being path-connected plays a role in the definition. The reason why path-connectedness is so important is made clear in the following theorem.

Theorem 3.4.2. *Let X be a path-connected space and $x, y \in X$ be any two distinct points. Then $\pi_1(X, x) \cong \pi_1(X, y)$.*

Proof. Since X is path-connected, let $g : I \to X$ be a continuous map with $g(0) = x$ and $g(1) = y$ and $g^{-1} : I \to X$ be the inverse path such that $t \mapsto g(1-t)$, $\forall t \in I$. Then we will define a map $\psi : \pi_1(X, y) \to \pi_1(X, y)$, by $[f] \mapsto \left[g^{-1} * f * g\right]$. That is we are mapping the classes of $\pi_1(X, x)$ to classes

in $\pi_1(X, y)$. We need to show that this is an isomorphism, to do this we will verify that $\psi([f] * [h]) = \psi([f]) * \psi([h])$ is a homomorphism and then show that the $\hat{\psi} : \pi_1(X, y) \to \pi_1(X, x)$ is an inverse for ψ. To verify that it is a homomorphism, $\psi([f] * [h]) = \left[g^{-1} * f * h * g\right] = \left[g^{-1} * f * g * g^{-1} * h * g\right]$, since $g^{-1} * g$ is clearly the identity loop. This means that $\left[g^{-1} * f * g * g^{-1} * h * g\right]$
$= \left[g^{-1} * f * g\right] * \left[g^{-1} * h * g\right]$, by the property that $[f * h] = [f] * [h] \Rightarrow \psi([f]) * \psi([h])$, and so this is a homomorphism. Now, let $\hat{\psi}$ be the function that maps $\left[g * f * g^{-1}\right]$ to $[f]$. Then to show that it is an inverse, $\hat{\psi}(\psi([f])) = \left[g * \psi([f]) * g^{-1}\right]$, where $\psi([f]) \in \pi_1(X, y)$ which means that $\hat{\psi}$ maps elements of $\pi_1(X, y)$ to $\pi_1(X, x) \Rightarrow \left[g * (g^{-1} * f * g) * g^{-1}\right] = \left[g * g^{-1} * f * g * g^{-1}\right] = [f] \in \pi_1(X, x)$. Lastly, we need to show that ψ is an inverse of $\hat{\psi}$. Consider $\psi(\hat{\psi}([k]))$, where $\hat{\psi}([k]) \in \pi_1(X, x)$ and $[k] \in \pi_1(X, y)$. $\psi(\hat{\psi}([k])) = \left[g^{-1} * \hat{\psi}([k]) * g\right] = \left[g^{-1} * (g * k * g^{-1}) * g\right] = \left[g^{-1} * g * k * g^{-1} * g\right] = [k] \in \pi_1(X, y)$ which shows that this is an isomorphism. $\square$

The big result to come out from this theorem is that since for a path-connected space, the fundamental groups are the same at either point, this means that the point in which we base our loops does not matter. Bringing it back to simply connected spaces, we have that at every point, the fundamental group is trivial. Since the space is path-connected, it can be said that the triviality of the fundamental group at one point implies every point because the base point does not matter. Now for an example.

Example. *Let us consider a circle, which inherits the Euclidean topology. Now suppose that there is a loop which goes around the circle once, we will represent this loop by $\delta = uvw$, where u, v, w can be observed in the figure 3.1. From w, we can only proceed by moving by a u or a w^{-1}, if we do proceed by a w^{-1}, then this means we are undoing the loop, and we would have uv. At this point we can only proceed by a w or a v^{-1}, and if we keep going back, this will take us to $\delta = 1$. This, of course, is the trivial loop, so the way we will want to proceed is forward.*

Hence, instead of going back at w with w^{-1}, we will go forward with u, giving $uvwu$. At u, we can proceed by v or u^{-1}, but we know what going back does in this process, so we proceed by v, and now follow up with a w. What we end up with is a loop $uvwuvw$ that goes around the circle twice. We can continue doing this as many times as we would like to, where at either a u, a v, or a w we either go forwards or backwards, and what we end up with is a loop represented by $(uvw)^n$, $n \in \mathbb{Z}$. Therefore, the loop $(uvw)^n$ goes around the circle n times, for $n \in \mathbb{Z}$, and hence $\pi_1(S^1) = \mathbb{Z}$.

I want to direct our attention to the notion of higher homotopy groups. It turns out that the fundamental group is just the case for something more or less general, i.e., the case where $n = 1$, and in fact there are groups for $n > 1$. I do not want to go into too much detail, but it is something that needs to be mentioned. In particular, a fairly accessible theorem regarding the nth homotopy group of an $n-$sphere is something that we will look at. But first, we will define a homotopy group.

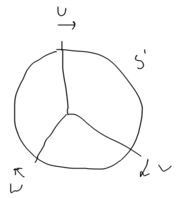

Figure 3.1: Going Around The Circle With u, v, w.

Definition 3.4.3. *Let* (X, x_0) *be a pointed space and* (S^n, s_0) *be the* n*-sphere with base point* s_0*. Then, the set of homotopy classes of continuous maps* $f : (S^n, s_0) \to (X, x_0)$*, denoted* $\pi_n(X, x_0)$ *admits an abelian group structure for* $n > 1$*, called the nth homotopy group of the pointed space* (X, x_0)*.*

Remark. *The concept of a pointed space is to give the space a distinguished point [9].*

It is clear from this definition that the generalization of n-dimensional homotopy groups happens by increasing the dimension on what we call a loop [3]. The way in which we defined the structure for homotopy groups can be given in a similar way as the fundamental group [3]. Even with this being the case I want to look at the way loops in a homotopy group compose. Consider two maps $f, g : S^n \to X$, then the way in which f and g compose as considered in [11]: we take a sphere with the base point s_0 on the equator, collapse the sphere to the base point s_0 so that we get two spheres at each end of the base point, and these two spheres get mapped into X by f and g see figure 3.2.

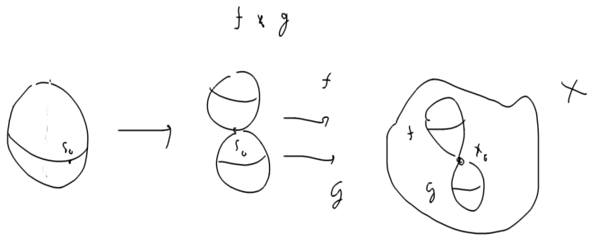

Figure 3.2: Mapping of The Sphere Into X

A rigorous proof of the following theorem is not something that we will do, but a proof can be found in [11]. However, I would like to point out that with a visualization of how loops compose, we do have a bit of insight as to why this theorem is true.

Theorem 3.4.4. *If $n > 1$, then $\pi_n(X, x_0)$ is abelian.*

The insight behind why this might be true is that if we compose two loops, f and g as we did in figure 3.2, at some point there are two spheres at either side of the base point. We can move the f sphere to the g sphere and the g sphere to the f sphere, which when mapped into X would give $g * f$. I want to give two theorems regarding the homotopy group of the n-sphere.

Theorem 3.4.5. *$\pi_m(S^n) = 0$, $\forall m, n$ with $0 < m < n$.*

This is just one case, we also have the following case as well,

Theorem 3.4.6. *$\pi_n(S^n) = \mathbb{Z}$, for every $n \geq 1$.*

Remark. *Interestingly, we do not necessarily find trivial homotopy groups when the dimension of the homotopy group is larger than that of the space.*

4 Homology

We have seen how the fundamental group is used to detect holes in a space. In the example of the circle, we found that the fundamental group is $\pi_1(S^1) = \mathbb{Z}$ meaning that S^1 has one hole. But, the fundamental groups only deals with 1-dimensional loops, and higher-dimensional homotopy groups use n-spheres as loops, which make computing these groups more difficult. So, we will turn to homology, which more or less, does the same thing, but in a different way.

We will begin by discussing simplicial homology, which deals with homology groups of polyhedra. At some point we will have to introduce the concept of a finitely generated abelian group. We will use this to make sense of both simplicial and singular homology. At the end of this section we will consider the Eilenberg-Steenrod axioms which makes use of properties of homology groups and category theory.

4.1 Simplices

Before we can do anything meaningful, we have to establish some of the terminology used throughout simplicial homology. As a preview, the basic set-up for what we are going to do is take a space, and divide it into geometric elements corresponding to verticies, edges, and faces of polyhedra [3]. More importantly, we will establish higher dimensional analogues of these features. So to start, we will consider what a p-dimensional simplex is. The best way in which we can understand this object is to treat a 0-simplex as a point; a 1-simplex as a line; a 2-simplex as a triangle; a 3-simplex as a tetrahedron; etc. So these simplices are just geometric objects of various dimension.

We want to be able to generalize the concept of a polyhedron to finite dimension, to do this, we will consider the following,

Definition 4.1.1. *Let σ_p be a p-dimensional simplex. If f is a simplex such that the vertices of f form a subset of the vertices of σ_p, then f is said to be a face of σ_p. A face f of σ_p is said to be proper if f is neither empty nor the entirety of σ_p.*

This is a nice way of generalizing the notion of a face to higher-dimensions. To see why, consider a 2-simplex, and suppose f is a 1-simplex. Then clearly the vertices of f form a subset of the 2-simplex because the vertices of the 1-simplex can be treated as vertices of the 2-simplex, and so the face of the 2-simplex is a line.

Definition 4.1.2. *A finite simplicial complex K is a finite collection of simplices in some Euclidean space $\mathbb{R}^n$ such that:*

i. If σ_p is a simplex in K, then all of its faces are also in K.

ii. If σ_p and σ_q are simplices in K, then $\sigma_p \cap \sigma_q$ is either empty or is a common face of σ_p or σ_q in K.

The next concept we introduce can be summarized as taking p-simplices, and giving them an ordering, we call this ordering an orientation.

Definition 4.1.3. *An oriented p-simplex σ_p for $p \geq 1$ is obtained from a p-simplex $\sigma_p = [v_0, v_1, \ldots v_p]$ by selecting an ordering of its vertices. An oriented simplicial complex is obtained from a simplicial complex by choosing an orientation of each of its simplices.*

4.2 The Simplicial Homology Group

Our goal will be to make a group out of these elements, for which we will have to introduce what is known as a finitely generate abelian group. This will allow for the discussion of a group whose elements are p-chains, which will be important soon. After that, we will consider boundary maps which more or less lowers the "dimension" of this group. We will see that these boundary maps are homomorphisms, and from here, we will be in a position to discuss the simplicial homology group. Lastly, we will look at the Euler characteristic and see how it is related to homology. Looking ahead into the next subsection, not much will change. Essentially, the only thing that will change is that we replace the notion of a simplicial complex by what we call a singular complex.

Definition 4.2.1. *An abelian group G is finitely generated if there exists elements $g_1, g_2, \ldots, g_n \in G$ such that every $g \in G$ can be written as,*

$$g = a_1 g_1 + a_2 g_2 + \ldots + a_n g_n, \text{ for } a_i \in \mathbb{Z}.$$

Remark. *The collection* $\{g_1, g_2, \ldots, g_n\} \in G$ *is called a basis for* G*.*

If G is an abelian group, an element $g \in G$ has finite order if $ag = 0$, for some positive $a \in \mathbb{Z}$. The set of all such elements is called the torsion subgroup of G, if the torsion group vanishes, then G is said to be torsion-free [12]. These are the abelian groups we will be working with, and are refered to as free abelian groups. For a free abelian group, we note the following,

Theorem 4.2.2. *If* G *is a nontrivial finitely generated abelian group with a basis of* n *elements, the* G *is isomorphic to* $\mathbb{Z} \times \mathbb{Z} \times \cdots \times \mathbb{Z}$*, for* n *factors.*

Remark. *This is the same as saying* $G \cong \mathbb{Z}^n$*.*

To see how this plays into what we are doing, let Ω be the set of oriented simplices on a simplicial complex K, in other words, Ω consists of elements from definition 4.1.3.

Definition 4.2.3. *Let* K *be a simplicial complex. A* p*-chain on* K *is a function* $c : \Omega \to \mathbb{Z}$*, such that:*

i. $c(\sigma) = -c(\sigma')$ *if* $\sigma, \sigma' \in \Omega$ *are the same simplex with opposite orientation.*

ii. $c(\sigma) = 0$ *for all but finitely many oriented* p*-simplices.*

The group resulting from adding these p*-chains is called the group of (oriented)* p*-chains of* K*, denoted* $C_p(K)$*. If* σ *is an oriented simplex, the elementary chain c corresponding to* σ *is the function defined as follows:*

$c(\sigma) = 1$*,*
$c(\sigma') = -1$*, if* σ' *is the opposite orientation of* σ*,*
$c(\tau) = 0$*, for all other oriented simplices.*

Remark. *For* $p < 0$*, the group of* p*-chains is trivial.*

The following lemma will give us a clear picture of how the concept of a free abelian group plays a role in what we are working toward.

Lemma 4.2.4. $C_p(K)$ *is a free abelian group; a basis for* $C_p(K)$ *can be obtained by orienting each* p*-simplex and using the corresponding elementary chains as a basis.*

Proof. Suppose all p-simplices of K are arbitrarily oriented. Then we can write each p-chain uniquely as a finite linear combination $c = \sum n_i \sigma_i$ of the corresponding elemenary chains σ_i. Now, c assigns to each oriented p-simplex, an n_i in the following way: σ_i gets assigned n_i, the p-simplex of opposite orientation gets assigned $-n_i$, and every other p-simplex not in the summation get assigned 0. We have found our basis, so this completes the proof. □

The linear combination that we used follows from the fact that for a free abelian group G, the general element of G can be written uniquely as a finite linear combination [12]. What we are left with, is essentially a group consisting of linear combinations of p-simplices. Now, we are going to see what we can do with this group. In particular, since this is a group, let us consider a group homomorphism, this will take the form of what is called the boundary operator.

Definition 4.2.5. *If $\sigma = [v_0, \ldots, v_p]$ is an oriented simplex with $p > 0$, we define $\partial_p \sigma = \sum_{i=0}^{p} (-1)^i [v_0, \ldots, \hat{v_i}, \ldots, v_p]$, where $\hat{v_i}$ denotes a vertex removed. Moreover, the map $\partial_p : C_p(K) \to C_{p-1}(K)$ is a homomorphism called the boundary operator, for which ∂_p is trivial for $p \leq 0$.*

Let us consider an example of how ∂_p is used.

Example. *Suppose, $\sigma = [v_0, v_1, v_2]$ is an oriented 2-simplex. Then $\partial_2 \sigma = [v_1, v_2] - [v_0, v_2] + [v_0, v_1]$. Furthermore, we note that this is a combination of 1-simplices, meaning that ∂_2 takes a 2-simplex to a 1-simplex.*

An important property of this operator can be found in the following lemma.

Lemma 4.2.6. $\partial_{p-1} \circ \partial_p = 0$.

Proof. Using how we have defined the boundary operator on an orineted simplex, we have, $\partial_{p-1}\partial_p [v_0, \ldots, v_p] = \sum_{i=0}^{p} (-1)^i \partial_{p-1} [v_0, \ldots, \hat{v_i}, \ldots v_p]$ which gives us, $\sum_{j<i} (-1)^i (-1)^j [\ldots, \hat{v_j}, \ldots, \hat{v_i}, \ldots] + \sum_{j>i} (-1)^i (-1)^{j-1} [\ldots, \hat{v_i}, \ldots, \hat{v_j}, \ldots]$ which cancels to give us zero. □

With the boundary operator defined, this allows for us to define the homology group.

Definition 4.2.7. *The group of p-cycles is the kernel of ∂_p, denoted $Z_p(K)$. The image of the ∂_{p+1}, called the group of p-boundaries, denoted $B_p(K)$. The quotient group $H_p(K) = Z_p(K)/B_p(K)$ is called the pth homology group.*

Remark. *By quotient group, I mean for $z \in Z_p(K)$, the group $H_p(K)$ has elements of the form $z + B_p(K)$.*

Remark. *Our previous lemma implies that $B_p(K)$ is a subgroup of $Z_p(K)$.*

Here is an example which emphasizes the homology group as a quotient group.

Example. *Suppose that we want to compute the first homology group, where we are given $ker(\partial_1) = \mathbb{Z}$ and $Im(\partial_2) = \{0\}$. Then we would compute the homology group by considering $ker(\partial_1)/Im(\partial_2) = \mathbb{Z}/\{0\}$. Using the remark accompanying the definition of the homology group, let $n \in \mathbb{Z}$, and we get something which looks like $n + \{0\}$. Meaning that for each n, we are getting a group element that belongs to the quotient group by adding it to the only element in $\{0\}$. In this case, we would find that $H_1 = \mathbb{Z}$.*

I want to turn our attention to the Euler characteristic. The Euler characteristic is a topological invariant, meaning it is the same for spaces that are homeomorphic. We will use homology here, but first, we will start by considering the rank of a finitely generated abelian group. Consider a free abelian group, that is, the finitely generated abelian group which is isomorphic to $\mathbb{Z}^n$. Then the rank of a free abelian group is just the number of copies of $\mathbb{Z}$, in this case the rank would be n. If there happens to be a torsion group, then we deal with this in the following way. As [12] suggests, if G is the finitely generated abelian group, T is the torsion group, and H is the free abelian group, then $G/T \cong H$, which gives us back the rank of a free abelian group.

To see where this discussion of rank is important in the context of homology, we will look at the Euler characteristic, denoted χ of a polyhedron. This can be given as $\chi = V - E + F$, where V denotes the number of vertices; E the number of edges; and F the number of faces. However, we can also express the Euler characteristic as $\chi = \sum_{p=0}^{n}(-1)^p \beta_p$, where β_p are the Betti numbers that we mentioned in our historical account of algebraic topology.

The following theorem, called the Euler-Poincaré theorem, will make use of the rank of a finitely generated abelian group by claiming that these Betti numbers are actually just the rank of the pth homology group. I will defer the proof because it makes use of stuff that we have yet to introduce, but a proof can be found in [3], however, I will give an example of how it is used.

Theorem 4.2.8. *The Euler characteristic of an oriented simplicial complex K of dimension n is given by $\chi(K) = \sum_{p=0}^{n}(-1)^p \beta_p = \sum_{p=0}^{n}(-1)^p rank(H_p(K))$.*

Example. *The homology group of a circle is given as $H_0(S^1) = \mathbb{Z}$, $H_1(S^1) = \mathbb{Z}$, and for $p > 1$ the homology group is trivial. This means that the rank of each nontrivial homology group is 1. Then the Euler characteristic for the circle is $\chi(S^1) = 1 \cdot 1 + (-1) \cdot 1$ (recall it is given as an alternating sum), so, it is not hard to see that $\chi(S^1) = 0$. If the space were instead S^2 (a sphere), we would find homology groups $H_0(S^2) = \mathbb{Z}$, trivial for $p = 1$, $H_2(S^2) = \mathbb{Z}$, and trivial for any $p > 2$. This makes the Euler characteristic, $\chi(S^2) = 1 \cdot 1 + (-1)^2 \cdot 1 = 2$.*

One thing that I want to emphasize is that the basic way of computing the Euler characteristic by looking at the number of vertices, edges and faces is limiting. There is more power behind using homology to do compute the Euler characteristic because we need not be restricted in this way, i.e., we can just use the homology group. Another important thing to remember is that the Euler characteristic is a topological invariant, so if a space is homeomorphic to the circle or sphere, then we know that it has the Euler characteristic 0 or 1, respectively.

4.3 Singular Homology

As we saw in the previous subsection, simplicial homology is limited to what sort of spaces we can apply it to, in other words, we are restricted to talking about polyhedra. To discuss homology in a more generic sense, we turn to singular homology. Singular homology has the advantage that we can discuss the homology group of a topological space X, and in this section we will establish this. In doing so, we just need to make some adjustments to simplicial homology. Mainly, we need to make changes to the nature of simplices, which will come in the form of a singular simplex. However, just because we are making adjustments to simplicial homology, does not mean that we are constructing the same homology, we are constructing something that is more natural compared to simplicial homology [12].

To start, we will define what the singular n-simplex is, however, we first need to introduce the standard singular simplex.

Definition 4.3.1. *Let $e_0, e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^{n+1}$. Then the standard n-simplex Δ_n is defined by,*
$\Delta_n = \{\sum_{i=0}^{n} \lambda_i e_i : \sum \lambda_i = 1,\ \textit{where}\ 0 \leq \lambda_i \leq 1\}.$

We can now use this in the following way. We consider the continuous map from the standard n-singular simplex to the topological space X. What we are doing here is essentially mapping a standard singular simplex onto the space.

Definition 4.3.2. *A singular n-simplex σ_n in a topological space X is a continuous map $\sigma_n : \Delta_n \to X$.*

We treat the singular 0-simplex as being identified with the points of X, the singular 1-simplex as being identified with the paths in X, etc [13]. With this definition, we can proceed with defining a chain group in in a similar manner as before,

Definition 4.3.3. *For every integer $n \geq 0$, the singular n-chain group $C_n(X)$ is defined as the free abelian group with basis on all singular n-simplices in X and $C_{-1}(X) = \{0\}$ is given.*

Remark. *An element of $C_n(X)$ is represented as a formal linear combination of singular n-simplices with integer coefficients and the default coefficient group for $C_n(X)$ is $\mathbb{Z}$ [3].*

Remark. *The author here uses ϕ to denote a singular 2-simplex instead of σ_2, and σ_2 to denote the standard 2-simplex instead of Δ_2.*

The boundary operator acts in the same manner as before, i.e., $\partial_n : C_n(X) \to C_{n-1}(X)$. We can look at faces of a singular n-simplex in the same sense as before, recall in definition 4.2.5, a vertex is removed by the boundary operator, and this carries over here. In addition, $Z_n(X) = \ker \partial_n$ and $B_n(X) = Im\partial_{n+1}$ are the singular n-cycles and singular n-boundaries, respectively. Hence the singular nth-homology group is given as,

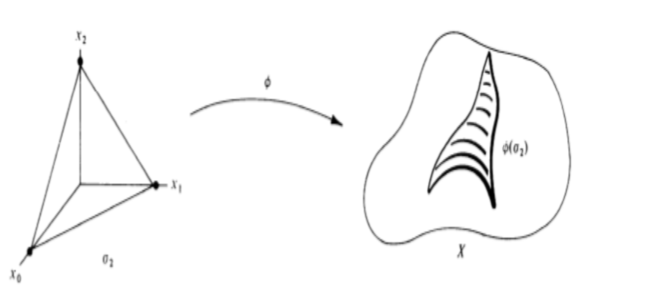

Figure 4.1: Mapping of a Standard 2-simplex Into X [13].

Definition 4.3.4. *Given a topological space X, for every integer $n \geq 0$, the nth singular homology group is defined as $\mathcal{H}_n(X) = \mathcal{Z}_n(X)/\mathcal{B}_n(X)$.*

Remark. *I've used calligraphic versions of what was used in the definition of the simplicial homology group to distinguish the singular homology group.*

4.4 The Chain Complex

Here, we will shift our attention to looking at some of the algebraic aspects of homology. Doing this will require introducing something called a chain complex. We will use this to look at homomorphisms between singular homology groups and look at the topological implications. I will remark that category theory will prove to be useful in our discussion.

We will begin by laying out the foundation that will be built upon throughout this section, and start by defining what a chain complex is.

Definition 4.4.1. *A sequence C_* of abelian groups $\{C_n\}$ and their homomorphisms $\{\partial_n\}$ as shown in figure 4.2 is said to be a chahin complex if $\partial_{n-1} \circ \partial_n = 0$, $\forall n \geq 1$.*

$$C_* : \qquad ... \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} ... \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} \{0\}$$

Figure 4.2: The Chain Complex.

There will be a subtle change in the terminology that we have used thus far, and we will give the new terminology below.

Definition 4.4.2. *Given a chain complex $C_* = \{C_n, \partial_n\}$, the homology group $H_n(C_*)$ is called the nth-homology group of the chain complex C_*. Z_n is called the group of n-dimensional cycles and B_n is the group of n-dimensional boundaries.*

An important feature in this approach is the notion of a chain map. We can think of this as a sequence of homomorphisms that map the chain groups C_n to the chain groups ${C'}_n$. The chain maps also have the property that they commute with the boundary homomorphisms ∂_n and ${\partial'}_n$. We will give this concept a formal definition.

Definition 4.4.3. *Given a two chain complexes $C_* = \{C_n, \partial_n\}$ and $C'_* = \{C'_n, \partial'_n\}$, a sequence $f = \{f_n : C_n \to C'_n\}$ of homomorphisms is said to be a chain map from C_* to C'_* if these homomorphisms commute with the boundary homomorphisms in the sense that $f_n \circ \partial_{n+1} = \partial'_{n+1} \circ f_{n+1}$, $\forall n \geq 0$ (every square in figure 4.3 commutes).*

$$\begin{tikzcd}
\dots \arrow[r] & C_{n+1} \arrow[r, "\partial_{n+1}"] \arrow[d, "f_{n+1}"] & C_n \arrow[r, "\partial_n"] \arrow[d, "f_n"] & C_{n-1} \arrow[r] \arrow[d, "f_{n-1}"] & \dots \\
\dots \arrow[r] & C'_{n+1} \arrow[r, "\partial'_{n+1}"] & C'_n \arrow[r, "\partial'_n"] & C'_{n-1} \arrow[r] & \dots
\end{tikzcd}$$

Figure 4.3: Diagram of The Chain Map.

We can observe, in the following proposition, how the n-cycles and n-boundaries will be affected under the chain map. The proof of which, as noted in [3] follows from the fact that each squrare in figure 4.3 commutes.

Proposition 4.4.4. *Given two chain complexes of abelian groups $C_* = \{C_n, \partial_n\}$ and $C'_* = \{C'_n, \partial'_n\}$ and a chain map $f = \{f_n : C_* \to C'_*\}$, the following is true:*

i. The images $f_n(Z_n) \subset Z'_n$ for each n, meaning f_n sends n-cycles of C_* to n-cycles of C'_*, and

ii. $f_n(B_n) \subset B'_n$ for each n, meaning f_n sense n-boundaries of C_* to n-boundaries of C'_*.

The next definition that we look at will introduce a homomorphism between homology groups. This idea will play a role in homotopic chain maps, which we will see after the proceeding definition.

Definition 4.4.5. *Given a two chain complexes $C_* = \{C_n, \partial_n\}$ and $C'_* = \{C'_n, \partial'_n\}$. The homomorphism $H_n(f_n) = f_{n^*} : H_n(C_*) \to H_n(C'_*)$ induced by the chain map $f : C_* \to C'_*$ is said to be the homomorphism induced by f_n in homology groups, for every n.*

If it is the case that two chain maps have a homotopy between them, then homotopic chain maps induce the same homomorphism in the homology.

Definition 4.4.6. *Given a two chain complexes $C_* = \{C_n, \partial_n\}$ and $C'_* = \{C'_n, \partial'_n\}$, with $f, g : C_* \to C'_*$ as their two chain maps. Then f, g are said to be chain homotopic, denoted $f \simeq g$, if there is a sequence $\{H_n : C_n \to C'_{n+1}\}$ of homomorphisms such that $\partial'_{n+1}H_n + H_{n-1}\partial_n = f_n - g_n : C_n \to C'_n$, $\forall n \geq 0$.*

With these definitions, we will see how we can use them, a proof of the following theorem can be found in [3].

Theorem 4.4.7. *Any two chain homotopic maps induce the same homomorphisms in the homology.*

We can use facts about category theory to establish something that will make proving how homeomorphic spaces are related in terms of homology groups, easier. The following theorem is taken from [3], it is just one of three in the list, but it is the one that is of most interest. I will also direct the reader to [3] for a proof of this theorem.

Theorem 4.4.8. *$H_n : \mathcal{T}op \to Ab$ is a covariant functor for each n, where $\mathcal{T}op$ is the category of topological spaces and their continuous maps, and Ab is the category of abelian groups and their homomorphisms.*

The following example makes it clear as to what the functor H_n is doing to the morphisms of $\mathcal{T}op$.

Example. *If $f : X \to Y$ is a continuous map in $\mathcal{T}op$, where X and Y are topological spaces, then $H_n(f) : H_n(X) \to H_n(Y)$.*

Finally, we arrive at something which is bigger than it first seems. Recall, most of what we are interested in doing in algebraic topology rests on the fact that homeomorphic spaces share the same algebraic invariant. Hence, it is necessary to mention the proceeding corollary.

Corollary 4.4.9. *Homeomorphic spaces induce isomorphic homology groups.*

Proof. We just need to show that if $X, Y \in \mathcal{T}op$ are homeomorphic, then the homology groups are isomorphic. Well, by theorem 4.4.8, the isomorphism just follows from the functorial property of
$H_n : \mathcal{T}op \to Ab$, for each $n \geq 0$. □

Without the functorial property argument, this corollary makes sense intuitively. If X and Y are homeomorphic, then it should stand that when we map singular n-simplices into the respective spaces, we do this in the same way for each (because they're homeomorphic). Thus, this should give rise to the same group structures for X and Y!

4.5 The Eilenberg-Steenrod Axioms

Constructing a homology theory is complicated [14]. We were able to define simplicial homology and singular homology somewhat easily, but to give all the properties, would be require a substantial amount of work [14]. To work around this, Eilenberg and Steenrod took the approach of considering a small number of properties of existing homology theories as axioms to characterize homology [3]. A homology theory assigns groups to topological spaces and homomorphisms to continuous maps of one space to another [14]. The axioms, then, are statements of the fundamental properties of the assignment of a group to a topological space, in addition, two such assignments give isomorphic groups [14].

We will begin by defining an admissible category for the axioms to be considered. Once this is done, we can introduce the Eilenberg-Steenrod axioms.

Definition 4.5.1. *A category C is said to be an admissible category for the Eilenberg-Steenrod axioms if C satisfies the following conditions:*

i. *C consists of topological pairs (X,A), as objects, i.e., if $(X,A) \in Ob(C)$, then the pairs $(X,X),(A,A),(X,\emptyset),(A,\emptyset),($ $Ob(C)$;*

ii. *C consists of morphisms $f:(X,A) \to (Y,B)$, for every pair of objects (X,A) and (Y,B) in $Ob(C)$ with all their possible inclusions, i.e., if $f:(X,A) \to (Y,B)$ is a morphism in C, then all inclusion maps, that are induced by f on the corresponding subpairs, are also morphisms in C;*

iii. *For any object $(X,A) \in Ob(C)$, the object $(X \times I, A \times I) \in Ob(C)$ and the maps $H_t : X \to X \times I$ are morphisms in C;*

iv. *There is the one-point space $X_0 \in Ob(C)$ with the property that the constant map $c : X \to X_0$ is a morphism in C.*

It is clear that all topological pairs and their continuous maps is an admissible category for the Eilenberg-Steenrod axioms, so we will keep this in mind when we proceed with consideration of the Eilenberg-Steenrod axioms. Before we do this, however, we will define a homology theory.

Definition 4.5.2. *A homology theory on the category $\mathcal{T}op^2$, denoted $\mathcal{H}$ is the triplet $\mathcal{H} = \{H, *, \partial\}$ which satisfies the following for every integer $n \geq 0$:*

i. *The function H assigns to every topological pair (X,A) of spaces in $\mathcal{T}op^2$ and every $n \geq 0$, an abelian group $H_n(X,A)$, called the n-dimensional homology group of the topological pair (X,A) in the homology theory $\mathcal{H}$;*

ii. *The function $*$ assigns to every continuous map $f:(X,A) \to (Y,B)$ in $\mathcal{T}op^2$ and every $n \geq 0$, a homomorphism $f_* : H_n(X,A) \to H_n(Y,B)$ called the homomorphism induced by f in the homology theory $\mathcal{H}$;*

iii. *The function ∂ assigns to each topological pair (X,A) in $\mathcal{T}op^2$ and every $n \geq 1$, a homomorphism $\partial : H_n(X,A) \to H_{n-1}(A)$ called the boundary operator on the group $H_n(X,A)$ in the homology theory $\mathcal{H}$.*

With everything necessary laid out, we now list the axioms.

Definition 4.5.3. *Let $\mathcal{H}$ be a homology theory on $\mathcal{T}op^2$. Then the functions of $\mathcal{H}$ satisfy the following axioms:*

i. *If $I_X : (X,A) \to (X,A)$ is the identity map on the topological pair (X,A) in $\mathcal{T}op^2$, then its induced homomorphism $I_{X^*} : H_n(X,A) \to H_n(X,A)$ is the identity automorphism of the homology group $H_n(X,A)$ for each $n \geq 0$;*

ii. *If $f:(X,A) \to (Y,B)$ and $g:(Y,B) \to (Z,C)$ are two continuous maps in $\mathcal{T}op^2$, then $(g \circ f)_{*n} = g_{*n} \circ f_{*n} : H_n(X,A) \to H_n(Z,C)$, for each $n \geq 0$;*

iii. *If $f : (X, A) \to (Y, B)$ is a continuous map in $\mathcal{T}op^2$ and if $g : A \to B$ is a continuous map in $\mathcal{T}op$ defined by $g(x) = f(x)$ for all $x \in A$, then we have commutativity in the sense that $g_* \circ \partial = \partial \circ f_*$;*

iv. *If (X, A) is a topological pair in $\mathcal{T}op^2$, and $i : A \hookrightarrow X$, $j : X \hookrightarrow (X, A)$ are inclusion maps, then the sequence found in figure 4.4 of groups and homomorphisms is exact, called the homology exact sequence of (X, A);*

v. *If two continuous maps $f, g : (X, A) \to (Y, B)$ in $\mathcal{T}op^2$ are homotopic in $\mathcal{T}op^2$, then $f_{*n} = g_{*n}$, $\forall n \geq 0$;*

vi. *Let U be an open set of a topological space X, where its closure $\overline{U}$ is contained in the interior of A, where A is a subspace of X. If the inclusion map $i : (X - U, A - U) \hookrightarrow (X, A)$ is in $\mathcal{T}op^2$, then the induced homomorphism $i_* : H_n(X - U, A - U) \to H_n(X, A)$ is an isomorphism for each $n \geq 0$. The inclusion i is called the excision of the open set U, and i_* is called the n-dimensional excision isomorphism;*

vii. *The n-dimensional homology group $H_n(X)$ of a one-point space X in the homology theory $\mathcal{H}$ consists of a single element for every $n \neq 0$, denoted $H_n(point) = \{0\}$, $for\ n \neq 0$.*

$$\cdots \to H_n(A) \xrightarrow{i_*} H_n(X) \xrightarrow{j_*} H_n(X, A) \xrightarrow{\partial} H_{n-1}(A) \to \cdots$$

Figure 4.4: Sequence of Inclusion Maps [3]

To summarize, if $\mathcal{H}$ is a homology theory, then these axioms must be satisfied. In particular, an example of Axiom 5 being satisfied in singular homology was when looking at the chain complexes, and if we recall, we said that homotopic chain maps induce a homomorphism in the homology.

5 Where Next

I want to wrap things up by looking at two theorems, one which I think is note worthy, and the other is the bigger picture of algebraic topology. We will not prove either of the theorems, but I want to be able to mention them. In particular, the Hurewicz isomorphism will point us in the direction in which the interest of algebraic topology may proceed.

The first theorem sort of parallels what we saw for homotopy groups of the n-sphere, this may be useful because homeomorphic spaces have isomorphic homology groups, so if our space is homeomorphic to the n-sphere, then we may rely on the following,

Theorem 5.0.1. *The homology group of an m-sphere is given by,*

$$H_n(S^m) = \begin{cases} \mathbb{Z} & if\ n = m \neq 0,\ or\ n = 0, m \neq 0 \\ \{0\} & otherwise \end{cases}$$

Something which may be unexpected is that homology groups and homotopy groups can be isomorphic. The first thing to note is that homology groups are abelian, and secondly, isomorphisms preserve structure. So, homotopy groups also have to be abelian, which we know for $n > 2$, this is the case.

Theorem 5.0.2. *Let $\pi_0(X, x_0) = \cdots = \pi_{n-1}(X, x_0) = 0$, where $n \geq 2$. Then $H_1(X, x_0) = \cdots = H_{n-1}(X, x_0) = 0$ and there is an $h : \pi_n(X, x_0) \to H_n(X, x_0)$ such that is an isomorphism.*

Remark. *We may not expect for this isomorphism to arise. However, it is worth mentioning, that comparing the homotopy group to the homology group of the m-sphere, we can observe that their structure is the same.*

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