POSSIBLE EFFECTS OF MODERN MATHEMATICS ON THE MATHEMATICS CURRICULUM OF NOVA SCOTIA

A thesis written in partial fulfilment of the requirements for the degree of Master of Arts.

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INTRODUCTION

The changes in mathematics at the present time are so extensive, so far reaching in their implications, and so profound that they can be described as a revolution.[1]

It is the object of this thesis to examine some of the causes of this revolution. The general public may be surprised to learn that mathematics is a live, active and growing subject. They may think that mathematics is complete and that there is no opportunity, need, or occasion for it to change. It is true that if a proposition is once true, it is always true. But propositions, like A-bombs, become obsolete as new and better ones are discovered.

The twentieth century has been the golden age of mathematics, insofar as the "new developments have been extensive . . . [and the] . . . bulk of current mathematical development is staggering."[2] This revolution now in progress makes us wonder whether the new mathematics should be taught in our schools, whether there should be a shift of emphasis in the teaching of many subjects already included in our mathematics courses, and an increase in the production of mathematicians and mathematics teachers.

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[1]See Floyd G.Robinson, "New Dimensions in Mathematics Teaching," C-I-L Oval, (August, 1961), p. 14.

[2]Program for College Preparatory Mathematics, Report of the Commission on Mathematics, (New York: College Entrance Examination Board, 1959), p.1.

The purpose of this thesis is threefold: to examine the present mathematics curriculum of the secondary schools in Nova Scotia, to consider in detail some of the new concepts and recent developments in mathematics, and finally to draw certain conclusions suggesting a revision in the approach to mathematics for high schools in this province.

In Chapter I the present mathematics curriculum in Nova Scotia from Grade IX to Grade XII will be discussed. The fact that the mathematics curriculum has changed very little in the past thirty years will be brought out, and an outline of the present mathematics curriculum of Nova Scotia is given in Appendix I for those not familiar with it.

Chapter II will outline the historical development of modern mathematics beginning with the discovery of non-Euclidean geometry. In addition, recently developed concepts and techniques of mathematics will be discussed.

In this connection, it should be observed that most, if not all, of the current programs to improve school mathematics avoid the presentation of new material as a string of unrelated topics. Indeed, they stress unifying themes or ideas in mathematics such as the following: extensive use of graphical representation, measurement, operations and their inverses, properties of numbers and the development of the real number system, language and elementary theory of sets, statistical inference and probability, structure, systems of numeration, valid generalizations, and logical deduction. An excellent example of this new approach to mathematics is seen in the Report of the Commission on Mathematics, a document studied in detail in Chapter III.

There have been several studies of mathematics made in Canada, but these have been on a relatively small scale, and detailed information on them is not readily available. With this in mind, the Commission on Mathematics Report, a representative study presenting recommendations for improvement of the mathematics curriculum without prescribing texts, was chosen for the case study in Chapter III. The Commission left it for others to write texts on its recommendations. Examination of the Commission findings will show that the Commission recommended revision of the present high school mathematics program to emphasize deductive reasoning in algebra, structure in mathematics, unifying ideas, treatment of inequalities, and incorporation of some coordinate geometry in the curriculum. A suggested sequence of topics for the high school curriculum is also included in the Commission's Report.

Chapter IV contains recommendations for improving the mathematics program in the secondary schools of Nova Scotia, recommendations based on the application of the Commission of Mathematics findings to the mathematics curriculum in this province.

CHAPTER I

MATHEMATICS IN NOVA SCOTIA

As this chapter will show, the mathematics curriculum in Nova Scotia has changed very little over the years. The texts and subject matter of the curriculum are outlined in the Nova Scotia Department of Education's *Program of Studies* and *Handbook to the Course of Study*. In addition there are two teaching guides; *Mathematics (Grades 7-9)* and *Teaching the New Geometry Courses in Grades 10 and 11*. An outline of the present mathematics curriculum in Nova Scotia is given in Appendix I, but for present purposes it is necessary to consider briefly each of the three branches of mathematics taught in Nova Scotian schools.

Algebra

In Nova Scotia a student begins algebra in Grade IX and continues the study of this subject through to Grade XII. As can be seen from the outline of the courses in Appendix I, there is considerable overlap in the curriculum of this subject from grade to grade.

The present Grade IX text was introduced in 1957, replacing a text that had been in use since 1935 and which covered much of the same material. The text for Grades X and XI has not changed since 1935. The text used in Grade XII after 1937 was replaced in 1957 with the present text. The new text, however, while it does not cover exactly the same material, exhibits the same basic approach to the subject matter.

Geometry[1]

A new geometry course was introduced into the schools of Nova Scotia during the 1960-1961 school year. The previous course was not too satisfactory, with most of the criticism of it levelled at the type of instruction given, rather than at the content of the course.

The main reason for teaching geometry is to give the students an understanding of the meaning of proof, that is, the proving of propositions from postulates, axioms and definitions. The best way to do this is by showing the students the various methods of proof, instead of solving the problem for them.

The members of the Nova Scotia Department of Education Mathematics Committee feel that the new text is more suitable for this type of teaching than the former text, and emphasize the importance of developing the students' initiative to solve problems.

In the new course more original work should be done by the students. Secondly, the proofs of all theorems are not required. Some theorems are to be accepted without proof, although the pupils should be shown that these appear to be true. Finally, the examinations should be such that a student who has done little more than memorize propositions from the book should not be able to pass.

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[1]The basis for this section is found in Robert Chafe, "Teaching of the New Geometry Courses in Grades 10 and 11," Education Office Gazette, (for Nova Scotia), March, 1961, pp. 57-58.

This new course should benefit all students to the extent that the new organization of subject matter provides increased opportunity for original thinking.

Trigonometry

Playne and Fawdry: Practical Trigonometry is used as a text in Nova Scotia for Grade XII. This text has been on the course of study since 1934. In Chapter III it will be shown that this type of course, from a mathematical point of view, is out of date, and for this reason no further description of it is offered at this point.

CHAPTER II

MODERN MATHEMATICS

The last few years have seen the launching of a program to effect basic change in the mathematics programs of both the elementary school and the secondary school. The principles or assumptions bringing about this change are twofold in nature: modern mathematics may be characterized by a new point of view and a new subject matter.

In this chapter the historical development of these points of view will be traced. Consideration will be given also to recently developed concepts and techniques of mathematics.

Modern Mathematics as a Point of View

The origin of what might be called the modern point of view in mathematics can be traced to the pioneering efforts of Carl Friedrich Gass (1777-1855), Johann Bolyai (1802-1860), Nicolai Ivanovitch Lobachevski (1793-1856) and Bernhard Riemann (1826-1866) in the creation of non-Euclidean geometries. It took unusual imagination to challenge what had been accepted as absolute for two thousand years, insofar as Euclid's geometry was widely accepted as the only true one.

The modern postulational method of mathematics finds its source in the publications by Lobachevski (1829) and Bolyai (1832) of a consistent geometric system which contradicted the Euclidean fifth postulate of parallelism while keeping all other postulates intact. It was further enhanced through the publication by Riemann (1854) of still another consistent non-Euclidean geometry based upon a still different contradiction of the fifth postulate. Subsequent refinements by Moritz Pasch (1843-1931), Guiseppi Peano (1858-1932), and David Hilbert (1862-1943) succeeded in establishing the purely hypothetico-deductive nature of geometry. In fact it has been said that Hilbert's work "firmly implanted the postulational method, not only in the field of geometry, but also in nearly every other branch of mathematics of the twentieth century."[1]

This new method no longer recognizes axioms or postulates as "self-evident truths," but merely as "acceptable assumptions." The choice of axioms is to some extent arbitrary and, in particular, they should not be regarded as self-evident statements of fact. Once this point of view is adopted the subject matter of the axioms is then best regarded as consisting of undefined elements, or rather as being defined only implicitly by the axioms. Taking this point of view mathematics may be regarded simply as an arbitrary creation of mathematicians, an attitude widely accepted today. The mathematician may become interested in the possibility of existence of physical or social applications which would provide a context of "truth" for his assumptions. The validity and consistency of results rather than the practicality of results, however, are his major concerns.[2]

The new method also places emphasis on the necessity for clear distinctions between what is defined and what must remain undefined. If we examine Hilbert's fifteen postulates for plane geometry,[3] it is immediately evident that certain basic terms are

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[1]Howard Eves and Carroll V. Newsom, An Introduction to the Foundations and Fundamental Concepts of Mathematics (New York: Rinehart and Company, Inc., 1958), p. 86.

[2]Cf., Dr. Paul R. Beesack, "Modern Mathematics Its Evolution, Logical Structure, and Subject Matter", New Thinking in School Mathematics, Report of a seminar held by the Canadian Teachers' Federation at Ottawa, April 28-30, 1960, p.73.

[3]Eves and Newsom, pp. 87-88.

completely undefined. A moment's reflection shows this must necessarily be the case if circularity in the definitions is to be avoided. Similarly, certain statements (axioms) involving basic terms must be left unproved. With this in mind what is regarded as the logical structure of a modern axiomatic system can be defined. It is a collection of statements beginning with some unproved statements (axioms, postulates) involving some undefined terms (basic terms, primitive terms), in which all further statements follow logically from the axioms and all new terms are defined in terms of the undefined or previously defined ones.[4]

Another factor, besides the discovery of non-Euclidean geometry, greatly influenced the development of the mathematical method. This was the recognition, first offered by the English mathematician George Peacock (1791-1858) about 1830, of the existence of structure in algebra. This recognition of algebra as an abstract science was developed further in England by Duncan Farquharson Gregory (1813-1844) and Augustus DeMorgan (1806-1871), and in Germany by Hermann Hankel (1839-1873). The central thought of this development had its origin in the recognition and abstraction of the fundamental properties which characterize the algebra of natural numbers (positive integers). Such abstraction provided the

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[4]M. Richardson, <u>Fundamentals of Mathematics</u> (New York: The MacMillan Company, 1958), p. 27.

symbolization which led to the realization that these same properties could very well characterize operations with elements other than the positive integers.[5] To illustrate this consider two binary operations, addition and multiplication, and if a, b, and c represent arbitrary natural numbers, distinct or not, then the properties may be stated as follows:[6]

Closure: If a and b are natural numbers, then $a+b$ and ab are unique natural numbers.

Associative: $a + (b + c) = (a + b) + c$

$a \times (b \times c) = (a \times b) \times c$

Commutative: $a + b = b + a$

$a \times b = b \times a$

Distributive: $a \times (b + c) = (a \times b) + (a \times c)$

Sir William Rowan Hamilton (1805-1865), a British mathematician, did for algebra what Lobachevski and Bolyai did for geometry. In 1843 he invented an algebra in which the commutative law of multiplication does not hold. In 1844 Germany's Hermann Grassmann (1809-1877) developed classes of algebras, and in 1857 the English mathematician, Arthur Cayley (1821-1895), devised matric algebra. Both of these developments were noncommutative

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[5]Eves and Newsom, pp. 120-121.

[6]For these laws see C. B. Allendoerfer and C. O. Oakley, Principles of Mathematics (Toronto: McGraw-Hill Book Company, Inc., 1955), pp. 40-45; Eves and Newsom, pp. 118-119; and Richardson, pp. 43-51.

algebras. Of more recent date are the nonassociative algebras such as those of C.Jordan (1838-1922) and M. S. Lie (1842-1899). Postulate sets for fields, rings, integral domains, and groups have evolved from this consideration of algebraic structure.

Another problem of major concern in any hypothetico-deductive system is the question of consistency. The meaning of 'consistency' is well expressed by Eves and Newsom in An Introduction to the Foundations and Fundamental Concepts of Mathematics.

> A postulate set is said to be consistent if contradictory statements are not implied by the set. This is the most important and most fundamental property of a postulate set; without this property the postulate set is worthless, and it is useless to consider any further properties of the set.
>
> The most successful method so far invented for establishing consistency of a postulate set is the method of models. A model of a postulate set, it will be recalled, is obtained if we can assign meanings to the primitive terms of the set which convert the postulates into true statements about some concept. There are two types of models-concrete models and ideal models. A model is said to be concrete if the meanings assigned to the primitive terms are objects and relations adapted from the real world, whereas a model is said to be ideal if the meanings assigned to the primitive terms are objects and relations adapted from some other postulate system.
>
> Where a concrete model has been exhibited we feel that we have established the absolute consistence of our postulate system, for if contradictory theorems are implied by our postulates, then the corresponding contradictory statements would hold in our concrete model. But contradictions in the real world we accept as being impossible.[7]

In 1868 E. Beltrami (1835-1900) brilliantly demonstrated:

> . . . that plane hyperbolic geometry can be interpreted as that of the geodesics of a surface of constant negative

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[7] Eves and Newsom, pp. 163-165

> curvature, and likewise for spherical geometry and a surface of constant positive curvature. Since pseudo-spheres and spheres are familiar surfaces in Euclidean space, it was felt that the consistency of the classical non-Euclidean geometries had been demonstrated.[8]

Through nineteenth-century researches by Peano, Dedekind and Cantor, the consistency of the real number system was shown to depend on the system of natural numbers, thus giving

> . . . the mathematician a considerable feeling of security concerning the consistency of most of mathematics. This attitude follows from the fact that the natural number system seems to have an intuitive simplicity lacking in most other mathematical systems, and the natural numbers have been very extensively handled over a long period of time without producing any known inner contradictions.[9]

The modern point of view in mathematics is described by Bell in these words:

> In precisely the same way that a novelist invents characters, dialogues, and situations of which he is both the author and master, the mathematician devises at will the postulates upon which he bases his mathematical systems. Both the novelist and the mathematician may be conditioned by their environments in the choice and treatment of their material; but neither is compelled by any extrahuman, external necessity to create certain characters or to invent certain systems.[10]

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[8] E. T. Bell, The Development of Mathematics (New York: McGraw-Hill Book Company, Inc., 1945), p. 332.

[9] Ibid., p. 195.

[10] Ibid., p. 330.

Modern Mathematics as New Subject Matter[11]

Modern mathematics as a new subject matter, like the modern point of view in mathematics, had its origin in the field of geometry. It has been said that the discovery of analytic geometry "marks a momentous epoch in the history of mathematical thought".[12] The analytic geometry of Pierre de Fermat (1601-1655, date of birth disputed) and René Descartes (1596-1650) remade geometry by breaking the hold of Greek classicism and making modern geometry possible. Though it was only a technique for geometric investigations, Cartesian geometry paved the way for modern mathematics. In 1637 Descartes published the work on which his greatness as a mathematician rests, his Discours de la Méthode, the third appendix of which contains his theory of analysis.[13] As a result of Descartes' innovation, analysis became the characterizing technique of modern mathematics, displacing classical geometry's emphasis on synthesis. This new geometry provided the basis for the differential and integral calculus of Isaac Newton (1642-1727) and G. W. Leibniz (1646-1716).

Fermat and B. Pascal (1623-1662) both published works in 1654

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[11] Bell, Chapter 7; and Appendices, Report of the Commission on Mathematics, (New York: College Entrance Examination Board, 1959) are the basis for this section.

[12] Alfred North Whitehead, An Introduction to Mathematics (New York: Oxford University Press, 1958), p. 81.

[13] Bell, p. 138.

which reduced chance to law, and in this way they became the founders of mathematical probability. These men were inspired by a gambling problem which Chevalier de Méré proposed to Pascal who, in turn, communicated it to Fermat. Each man solved it correctly, but by different reasoning. Thus the mathematics of chance originated.

Three other developments in mathematics occurred in the seventeenth century: G. Desargues (1593-1662) and Pascal built a good foundation for synthetic projective geometry; Pascal at nineteen invented the adding machine, an instrument improved by Leibniz who extended its operations to multiplication; and Leibniz also laid the foundation for modern symbolic logic.

So great were the significance and impact of coordinate geometry that the new projective geometry of Desargues and Pascal lapsed into temporary oblivion. There it remained until revived by the publications of L. N. M. Carnot (1753-1823) and J. V. Poncelet (1788-1867) in 1803. Mathematical investigation was given further incentive in 1825-1827 with the announcement by J. D. Gergonne (1771-1859) of the principle of duality "which, with its generalizations, left as substantial a residue of new and useful methods, in geometry, algebra, and analysis as any mathematical invention of the nineteenth century."[14]

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[14] Bell, p. 341.

In 1872 Felix Klein (1849-1925) announced his Erlanger program. In an address delivered on acceptance of a chair at the University of Erlanger he incorporated a definition of geometry which appeared to restore order to the confusion existing in this science. He defined a geometry as "the system of definitions and theorems which express properties invariant under a given group of transformations."[15]

Probably the most significant and most basic of all the newer concepts of modern mathematics is that of set.[16] Set (aggregate, ensemble, assemblage, family, or class) is an undefined concept. However, a set can be thought of as a well-defined collection of objects. There are two different ways of designating a set. One can give a complete list of all the elements of the set or one can give a rule by which it is determined whether or not an object is an element of the set. The former is designated a _listing of the set_, the latter a _description of the set_.

It is customary to use a capital letter for the name of a set, small letters for its elements, and to use braces to surround

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[15] Oswald Veblen and John Wesley Young, _Projective Geometry_, II (Boston: Ginn and Company, 1918), p.71. See also Eves and Newsom, p. 135.

[16] For an excellent discussion of some of the basic concepts of set theory see John G. Kemeny, J. Laurie Snell, and Gerald L. Thompson, _Introduction to Finite Mathematics_, (Englewood Cliffs, N.J. Prentice Hall, Inc., 1956), Chapter II. Cf., Allendoerfer and Oakley, Chapter 5.

the listing of a set. For example: $A = \{1, 2, 3, 4, 5\}$ or $B = \{x$, such that $x < 2\}$. The second set can also be written $B = \{x / x < 2\}$, and it is read: "B is the set of x's such that $x < 2$." The vertical bar, "/" is read: "such that."

If all the members of a set A are also members of a set B, A is called a <u>subset</u> of B. If B has members that are not in A, then A is a <u>proper subset</u> of B. Sometimes we are interested in one or more subsets of one overall set called the <u>universal set,</u> U. The <u>empty set</u>, $\emptyset$, is one that has no members. For example, the set of all perfect squares that end in 2, 3, 7 or 8 is $\emptyset$. The <u>unit set</u> is a one element set, e.g. the set of all even one-digit prime numbers is $\{2\}$.

The <u>intersection</u> of two sets A and B is the set composed of those elements that are in both A and B. The symbol for intersection is $\cap$. $A \cap B$ is read: "the intersection of A and B," or "A cap B."

The <u>union</u> of two sets A and B is the set that contains those and only those elements that belong either to A or to B (or to both). The symbol for union is $\cup$. $A \cup B$ is read: "the union of A and B," or "A cup B."

If A is a given subset of the universal set U, we can define a new set A' called the <u>complement</u> of A as follows: A' is the set of all elements of U that are not contained in A.

$$A' = \{x / x \text{ is not a member of } A\}$$

Sets can be defined by equations. The equation $x - 5 = 0$ can be used to define the set $\{x \,/\, x - 5 = 0\} = \{5\}$. Similarly, $\{x/\, x^2 - 6x + 5 = 0\} = \{1, 5\}$ is defined by the equation $x^2 - 6x + 5 = 0$. In general, any equation involving x defines a set consisting of those values of x which satisfy the equation. Hence we may speak of the <u>set of solutions</u> of an equation, or its <u>solution set</u>.

The Nature of Deductive Methods

One of the most significant of all the new emphases affecting the modern program in mathematics directs attention to the axiomatic structure of mathematics. From the point of view of logic, the number of properties assumed should be minimized. Such a procedure maximizes the number to be proved. It is desirable at times, however, to relax this criterion in order to present an equivalent structure more readily comprehensible to the immature mind.

For example, a postulational basis for the natural numbers was first announced in 1889 by the Italian mathematician G. Peano (1858-1932). Eves and Newsom state Peano's theory as follows:

> For primitive terms Peano chose natural number, successor, and 1. About these primitive terms he postulated:
> N'1: 1 is a natural number.
> N'2: For each natural number x there exists exactly one natural number, called the successor of x, which will be denoted by x'.
> N'3: 1 is not the successor of any natural number.
> N'4: if x'=y', then x=y.
> N'5(the postulate of finite induction): Let M be a set of natural numbers such that
> (1) M contains 1,
> (2) M contains x' whenever it contains x, then M contains all the natural numbers.[17]

From Peano's postulates all the properties of the natural numbers can be deduced and, through proper extensions, the properties

[17] Eves and Newsom, p. 203.

of all integers, rational numbers, real numbers, and complex numbers can also be deduced. A less sophisticated, but equivalent, set of postulates which is more easily comprehended by the immature student is the following example from Eves and Newsom:

> For our primitive, or undefined, terms we take a set N of elements called natural numbers, together with two binary operations on the set, called addition and multiplication and denoted by + and x, satisfying the following ten postulates.
>
> N1: If a and b are in N, then $a+b=b+a$.
>
> N2: If a and b are in N, then $a \times b=b \times a$.
>
> N3: If a, b, c are in N, then $(a+b)+c=a+(b+c)$.
>
> N4: If a, b, c are in N, then $(a \times b) \times c=a \times (b \times c)$.
>
> N5: If a, b, c are in N, then $a \times (b+c)=(a \times b)+(a \times c)$.
>
> N6: There exists a natural number 1 such that $a \times 1=a$ for all a in N.
>
> N7: If a, b, c are in N and if $c+a=c+b$, then $a=b$.
>
> N8: If a, b, c are in N and if $c \times a=c \times b$, then $a=b$.
>
> N9: For a given a and b in N, one and only one of the following holds: $a=b$, $a+x=b$, $a=b+y$, where x and y are in N.
>
> N10: If M is a set of natural numbers such that (1) M contains the natural number 1, (2) M contains the natural number $k+1$ whenever it contains the natural number k, then M contains all the natural numbers.[18]

The relation of equality between two or more natural numbers in the above postulates is characterized by the following properties:

E1. Reflexive Property. If a is in N, then $a=a$.

E2. Symmetric Property. If a, b are in N and $a=b$, then $b=a$.

E3. Transitive Property. If a, b, c are in N such that $a=b$ and $b=c$, then $a=c$.

E4. Additive Property. If a, b, c are in N and $a=b$, then $c+a=c+b$.

E5. Multiplication Property. If a, b, c are in N and $a=b$, then $ca=cb$.

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[18] Eves and Newsom, p. 195.

The first five postulates (N1-N5) are exact counterparts of the first five postulates for the real number system.[19] They postulate the commutativity and associativity of the two binary operations of addition and multiplication and the distributivity of multiplication over addition. N6 postulates the existence of a multiplicative identity, and N7 and N8 grant the cancellation laws for addition and multiplication. N9 introduces the idea of order among the natural numbers. This leads to three basic relationships between any two natural numbers a and b: $a=b$ (a equals b), $a > b$, (a is greater than b), or $a < b$ (a is less than b). N10 is known as the postulate of finite induction and leads to the theorem which is referred to as the principle of mathematical induction.

The five properties (E1-E5) characterize the relationship of equality between natural numbers. They also hold for equality between all numbers. The first three properties (E1-E3) characterize the more general equivalence relation.

From the ten postulates of the natural number system and the five characteristic properties of equality all the remaining properties of the natural number system can be deduced as theorems. (Definition of terms along the way will be necessary.)

For example, consider the often quoted axiom of multiplication: if equals are multiplied by equals, the products

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[19] _Supra_, p. 13.

are equal. We may now state it and prove it as follows:

Theorem: If a, b, c, d are in N and if a=b and c=d, then ac=bd.

Proof:	(1) The product of any two of the elements a, b,c,d is in N.	(1) Closure property.
	(2) a=b	(2) Hypothesis.
	(3) ac=ca	(3) N2
	(4) ca=cb	(4) E5 and hypothesis.
	(5) cb=bc	(5) N2
	(6) c=d	(6) Hypothesis.
	(7) bc=bd	(7) E4
	(8) Therefore ac=bd	(8) Steps 2,3,4,7 and E3.

One of the simplest of all significant algebraic structures is that of a group. The idea of a group is due to E. Galois (1811-1832), who died in a duel at the age of twenty. Its structure follows:

Undefined elements: a, b, c, . . . belonging to a set G.

Undefined operation: o, used to pair two elements: a o b.

Axioms:

G1. For each ordered pair, a and b in G, the combination a o b is a unique element c of G. (Axiom of closure.)

G2. For each triple, a, b, and c in G:

$$(a \circ b) \circ c = a \circ (b \circ c)$$

(Associative axiom.)

G3. There exists a unique element e of G having the property that for every a in G:

$$a \circ e = e \circ a = a$$

The element e is called the identity.
(Identity axiom.)

G4. Corresponding to each a in G there is a unique element a' having the property that

$$a \circ a' = a' \circ a = e$$

The element a' is called the inverse of a. (Inverse axiom.)

G5. For every a and b in G,

$$a \circ b = b \circ a$$

(Commutative axiom.)

A group which also satisfies axiom G5 is called a commutative group. A group will not be assumed to be commutative unless we specifically say that it is.[20]

As an example of a finite group, the group of "symmetries" of an equilateral triangle can be considered.[21]

The symmetries of the triangle are motions that bring it into coincidence with itself. We shall consider two motions as being the same if they have the same effect. It is easily seen that this will result in a system consisting of six elements, I (no motion), P (rotate 120° clockwise), Q (rotate 240° clockwise), R (flip over, keeping top vertex fixed), S (flip over keeping left vertex fixed), and T (flip over keeping the right vertex fixed).

Let us define a binary operation for this system as follows: If A and B represent any two of these motions, the product A o B is the motion that results when the two motions are performed one right after the other, with B performed first, and A taking over where B

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[20] Allendoerfer and Oakley, p. 71.

[21] This illustration is found in Irving Adler, The New Mathematics, (New York: The John Day Company, 1958), pp. 55-60; and for a similar illustration but with a square see George A. W. Boehm and the Editors of Fortune, The New World of Math, (New York: The Dial Press, 1959), pp. 24-25.

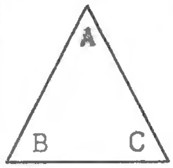
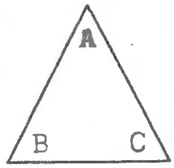
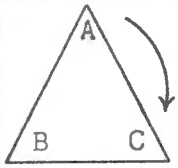
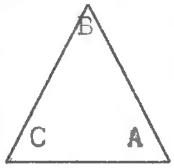
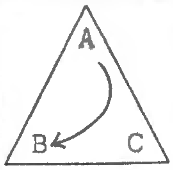
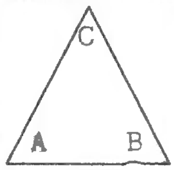
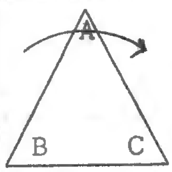
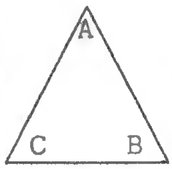
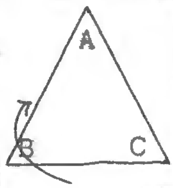
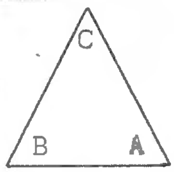
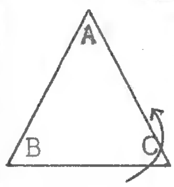
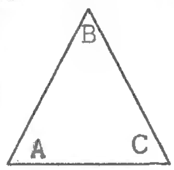
Motion	Symbol	First Position	Final Position
No motion	I		
Rotate 120° clockwise	P		
Rotate 240° clockwise	Q		
Flip over, keeping top vertex fixed	R		
Flip over, keeping left vertex fixed	S		
Flip over, keeping right vertex fixed	T		

Fig.1.--Motions of the triangle

o	I	P	Q	R	S	T
I	I	P	Q	R	S	T
P	P	Q	I	S	T	R
Q	Q	I	P	T	R	S
R	R	T	S	I	Q	P
R	S	R	T	P	I	Q
T	T	S	R	Q	P	I

Fig.2.--Multiplication table for symmetries of the triangle

leaves off. If we pick any two of the six motions at random, and we perform one right after the other, we find that the result is always one of the original six motions. The results of performing the various operations can be summarized in the multiplication table (Figure 2), where the motion performed first and written on the right side in a product is listed at the top of the table, and the motion performed second and written on the left side in a product is written at the left side.

To show that the symmetries of the triangle form a group, we have to prove that the three requirements for a group are satisfied. (1) The operation o is associative. This can be verified from the table. (2) It is obvious from the table that I is the identity element. (3) Every element has an inverse element for $P \circ Q = Q \circ P = I$, $I \circ I = I$, $R \circ R = I$, $S \circ S = I$, and $T \circ T = I$. Therefore the symmetries of the triangle, with operation o as defined, form a group.

The concept of a field is one of the most important of all the concepts of algebra. A field is an abstract mathematical system given by the following:

Undefined terms:
Elements: a, b, c, . . . of a set F. We assume that at least two elements exist.
Operations: +; x. The product "axb" will be frequently written "ab" or "(a) (b)."
Axioms:
R1. The sum of a+b of each pair of elements of F is a unique element c of F. (Closure.)
R2. For any triple of elements of F, $(a+b)+c = a+(b+c)$ (Associative law.)

R3. There exists a unique element of F, called zero, such that for every element a of F

$$a+0 = 0+a = a$$

The element zero is called the additive identity of F. (Existence of zero.)

R4. Corresponding to each a of F there is a unique element -a in F such that $a+(-a) = (-a)+a = 0$

The element (-a) is called the additive inverse of a. (Existence of additive inverse.)

R5. For every pair of elements of F,

$$a+b = b+a$$

(Commutative law.)

R6. The product axb of each pair of elements of F is a unique element of F. (Closure.)

R7. For any triple of elements of F,

$$(a \times b) \times c = a \times (b \times c)$$

(Associative law.)

R8. There exists a unique element of F, called the unit element and written 1, such that for every element a of F,

$$a \times 1 = 1 \times a = a$$

(Existence of the unit element.)

R9. Corresponding to each a of F (except zero) there is a unique element 1/a in F such that

$$a \times \frac{1}{a} = \frac{1}{a} \times a = 1$$

The element 1/a is called the multiplicative inverse of a. (Existence of multiplicative inverse.)

R10. For every pair of elements of F,

$$a \times b = b \times a$$

(Commutative law.)

R11. For every triple of elements of F,

$$a \times (b+c) = (a \times b)+(a \times c)$$

(Distributive law.)[22]

One or two simple illustrations will point up the basic significance of the concept of field. In the field of complex numbers both $x^2-y^2=(x-y)(x+y)$ and $x^2+y^2=(x+iy)(x-iy)$ are factorable, while in the field of real numbers only x^2-y^2 is factorable. The equation $x^2+1=0$ has no solution in the field of real numbers but has two

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[22]Allendoerfer and Oakley, pp. 83-85.

solutions, i and -i, in the field of complex numbers. In other words, in the field of real numbers the solution set for $x^2+1=0$ is the null set $\emptyset$, while in the field of complex numbers it is $\{i, -i\}$.

An interesting example of finite geometry as a postulational system which exhibits absolute consistency can be constructed in the following manner:[23]

As undefined concepts we take the following and indicate the concrete notions from which these have been abstracted:

Undefined Concepts	Concrete Illustrations
A set of elements A, B, C, belonging to a class S	A set of points in a plane
m-class	a line
belonging to an m-class	a point lies on a line

Use as the axioms of this system the following statements:

FG1. If A and B are distinct elements of S, there is at least one m-class to which both A and B belong.

FG2. If A and B are distinct elements of S, there is not more than one m-class to which both A and B belong.

FG3. Any two m-classes have at least one element of S in common.

FG4. There exists at least one m-class.

FG5. Every m-class contains at least three distinct elements of S.

[23]Veblen and Young, I, pp. 1-7.

FG6. All the elements of S do not belong to the same m-class.

FG7. No m-class contains more than three distinct elements of S.

Before considering theorems in this system, let us first check to see that the axioms are consistent. Consider the array:

A	B	C	D	E	F	G
B	C	D	E	F	G	A
D	E	F	G	A	B	C

If the columns in this array represent m-classes, we see that this array satisfies all seven axioms. Hence the axioms are consistent.

Let us consider some other interpretions of this system:

Element	m-class	Belongs to
person	committee	is a member of
diameter of a sphere	great circle of a sphere	the diameter is the diameter of the great circle
member of a lunch club	luncheon	ate at a particular luncheon.

Each of these can be a concrete representation of our abstract system if they satisfy all the conditions, and the theorems stated below must then apply. These theorems are not hard to prove, and give good practice in the deductive method.

FGT1. Any two distinct elements of S determine one and only one m-class containing both these elements.

FGT2. Any two m-classes have one and only one element of S in common.

FGT3. There exist three elements of S which are not all in the same m-class.

FGT4. Any class S satisfying these axioms contains at least seven elements.

While the above postulational system is orientated to geometry as to content, it can serve as a mathematical model for problems in distinctly different contents. Consider the following problem. What is the minimum number of airplanes necessary to stage a demonstration designed to meet the following specifications? Two planes are to fly together in one and only one formation. For any two formations there must be at least one plane in each of them. Exactly three planes are to fly in each formation. All planes cannot fly in the same formation.

If "plane" is identified with "element", "formation" with "m-class", and "flies on" with "belonging to an m-class", then it is evident that the specifications of this problem can be identified with FG1-7. Therefore, the solution to the problem is given in FGT4 as seven, and the formation is given in the following figure.

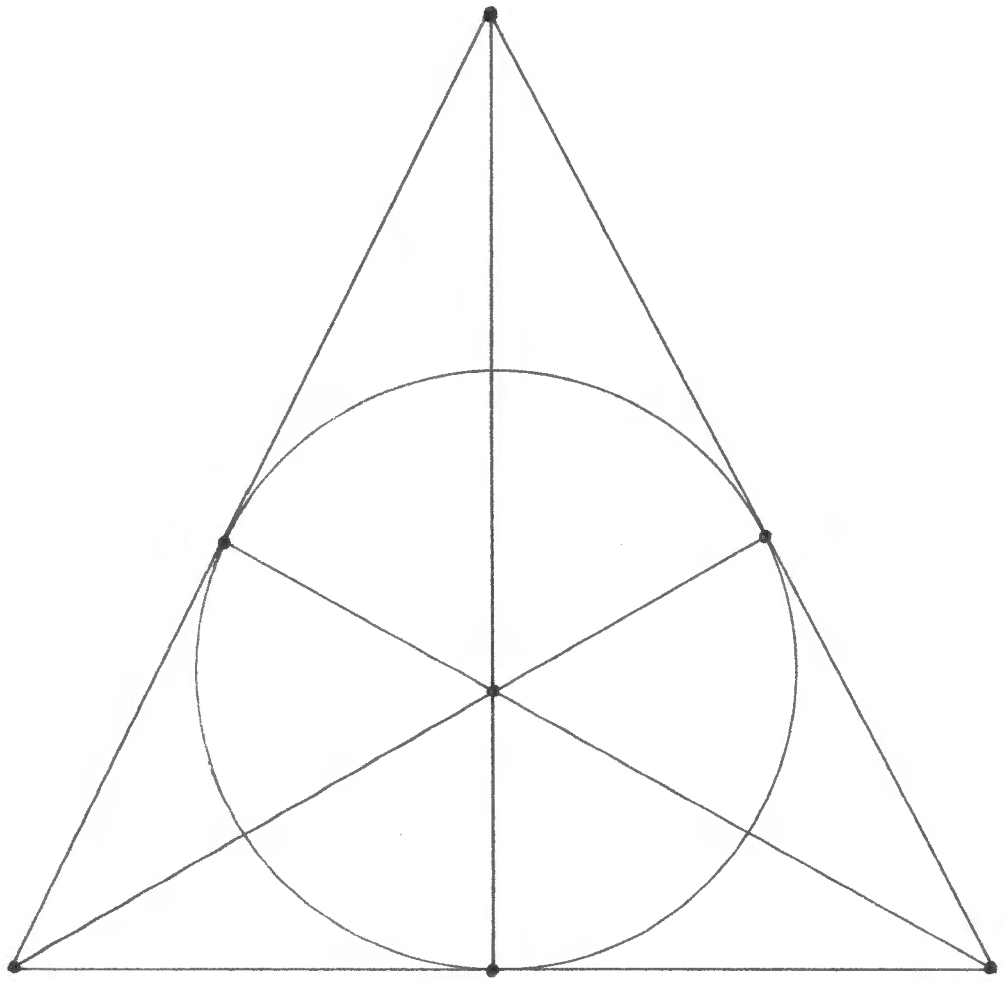

Fig. 3.--The formation of the planes

Techniques of Deduction[24]

The new emphasis on the axiomatic structure of mathematics emphasizes the need for clearer understanding of the valid techniques of deductive thinking. This is essentially the combination of statements or propositions into still other propositions. There are five basic connectives in deduction, and the following table expresses them symbolically.

Name	Symbol	Translated as
Conjunction	$\wedge$	"and"
Disjunctive	$\vee$	"or"
Negation	$-$	"not"
Implication	$\rightarrow$	"if...then..."
Equivalence	$\leftrightarrow$	"...if and only if..."

A set of rules for the proper use of these connectives in valid deduction is called a truth table, examples of which are given below. The symbols p and q represent propositions, a sentence so clearly stated that it can be declared unequivocally to be true or false. The letter T indicates that the proposition whose symbol is at the top of the column is true, while F designates a proposition as false in a similar manner.

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[24] For a discussion of such techniques see Kemeny, Snell, and Thompson, Chapter 1; Allendoerfer and Oakley, Chapter 1; or Henry W.Johnstone, Jr., Elementary Deductive Logic, (New York: Thomas Y. Crowell Company, 1954), Parts one and two.

p	q	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$	$-p$
T	T	T	T	T	T	F
T	F	F	T	F	F	F
F	T	F	T	T	F	T
F	F	F	F	T	T	T

One of the most important principles for the process of deductive reasoning is the law of the syllogism: If the proposition p implies the proposition q and the proposition q implies proposition r, then proposition p implies proposition r. Symbolically, $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$. This proposition is a **tautology**. That is, it is true regardless of whether its component propositions are true or false. Its truth table follows:

p	q	r	$p \rightarrow q$	$q \rightarrow r$	$(p \rightarrow q) \wedge (q \rightarrow r)$	$p \rightarrow r$	$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

From a given implication, $p \rightarrow q$, we can form a number of related implications which may or may not be true even if the given implication is true. Important ones are:

Converse: $q \rightarrow p$

Inverse: $(-p) \rightarrow (-q)$

Contrapositive: $(-q) \rightarrow (-p)$

The following truth tables lead to these conclusions.

(1) The converse of a true implication is not always true.

(2) The inverse of a true implication is not always true.

(3) An implication and its contrapositive are simultaneously true or false; i.e., they are equivalent.

p	q	-p	-q	Proposition $p \rightarrow q$	Converse $q \rightarrow p$	Inverse $(-p) \rightarrow (-q)$	Contrapositive $(-q) \rightarrow (-p)$
T	T	F	F	T	T	T	T
T	F	F	T	F	T	T	F
F	T	T	F	T	F	F	T
F	F	T	T	T	T	T	T

Much can be said about the techniques of deduction; the foregoing merely points out some of its fundamental pattern and basic principles.

That the logical character of deduction can be expressed as an algebraic structure is,however, quite clear. And in such a form of expression the following elements of structure are basic:

The undefined terms and their symbols are:

proposition	p, q, . . .
true	T
false	F

and	$\wedge$
or	$\vee$
implies	$\rightarrow$
equivalent	$\leftrightarrow$
not	$-$

The defined terms are:

Conjunction	$p \wedge q$
disjunction	$p \vee q$
implication	$p \rightarrow q$
equivalence	$p \leftrightarrow q$
negation	$-p$

The axioms are:

(1) Every proposition is either true or false, but not both true and false.

(2) The expressions given by our defined terms above are propositions.

(3) The truth tables for conjunction, disjunction, implication, equivalence, and negation.

Methods of Proof

Mathematics is primarily a deductive science in that propositions are proved by showing that they are implied by propositions already proved, definitions already stated, and axioms already accepted. The definitions of fundamental terms should be clear, simple, and brief. And in addition normally there will be terms accepted as established elements of common knowledge. For example, there is no clarification of concepts gained by defining point, line and plane in Euclidean geometry.

Similarly, there will be a list of axioms. It should be emphasized that these axioms are merely statements accepted as true because of their conformity with common experience and sound judgement and in no case should they be considered as "self evident truths." The principal characteristics of a set of axioms are: Consistency; there should be no contradictory statements in the list. Simplicity of statement; the axioms should be free from ambiguous statements and should be in a form that will permit ready deductions. The axioms should present no conflict with established knowledge or observable facts.

When a mathematical system has been clearly structured by the selection of the undefined elements, the definition of basic terms, and the acceptance of a set of axioms, then the proving of theorems becomes the major concern. Every theorem has two characteristic properties,a hypothesis and a conclusion.

The hypothesis is a statement, simple or compound, of the accepted relationships existing between the elements of the given structure which are to be used in the search for the new relationships which are summed up in the conclusion, again a simple or compound statement. The proof of the theorem consists in the establishment of the truth of the conclusion through implications and inferences that find their original source of justification in the hypothesis. There are three distinct processes to be used in establishing the proof of any given theorem.

Synthetic Process The synthetic process consists of drawing a series of necessary conclusions until the desired conclusion is reached. Although this process is simple and elegant, it makes no provision for the pupil to understand the reason for making significant constructions or applying auxiliary theorems.

Analytic Process The analytic process considers the desired conclusions and reasons that they are implied by the conclusions of some previous proposition. One follows such a chain of reasoning back to the hypothesis of the unproved theorem, thus closing the logical chain necessary to establish the validity of the desired conclusion.

Analytic-Synthetic Process The analysis is the process of discovering ways and means of arriving at desired results. The investigator considers the desired conclusion and raises the question: "What relation or property is sufficient to justify the use of this conclusion as a true statement?" Once this relation is found, he analyses it for the same purpose, with the hope in mind that finally he will arrive at the hypothesis of the theorem as the source of the chain of sufficient reasons. This process does not constitute a proof, however, until it has been established that the steps are reversible. The reverse argument is the synthesis. This analytic-synthetic process is an effective technique in guiding the immature student in the ways of discovery and validation of results.

The two most important types of proof are direct proof and indirect proof. In a direct proof one starts with the hypothesis and proceeds through a chain of syllogistic reasoning to the implication of the desired conclusion.

When we are unable to find a direct proof of a proposition, we frequently turn to the much misunderstood method of indirect proof. This method relies on the fact that if (-p) is false, then p is true. Hence to prove that p is true, we attempt to show that (-p) is false. The best way to accomplish this is to show that (-p) is not consistent with the given proposition. In other words, we add (-p) to the list of given propositions and attempt to show that this augmented set of propositions leads to a contradiction. When the contradiction is reached, we know that (-p) is not consistent with our given true propositions and hence that it is false. Hence p is true. Indirect proof is a method of reaching a desired conclusion through the process of investigation and elimination of all other mutually exclusive possibilities.

Two other forms of proof are the existence proof and the enumeration proof. The existence proof consists of setting up an example which establishes the truth of the proposition. The enumeration proof consists of checking each case for the truth of a proposition. This latter method of proof will work only when the number of cases is finite and reasonably small.

A counterpart to proof is, of course, disproof, and the utility of a method of disproof lies in the fact that an unsuccessful attempt to prove a conjectured theorem leads naturally to the countering attempt to disprove it. One of the usual methods of disproof is to assume that the theorem is true and then derive consequences from this. If we succeed in arriving at a consequence which contradicts a known true theorem, we have shown that the conjectured theorem is false. This process is so similar to the method of indirect proof that no further remarks need be made about it.

The use of counterexamples is another effective method for disproving statements. For example, let m and n be arbitrary odd numbers. Then m+n is an odd number. The disproof is immediate from the counterexample $1+5=6$.

A final point regarding disproof which is important to remember is that, although disproof by a single example is a valid method of procedure, theorems can be proven by considering particular cases only when the number of these is sufficiently finite that they can all be tested.

Discussion of methods of mathematical proof would be incomplete without some reference to induction, the process of discovering general laws by the observation and combination of particular instances. This method is used in all sciences, even in mathematics. Mathematical induction is used in mathematics alone to prove theorems of a certain kind. It is rather unfortunate that the word 'induction' is used of both mathematics and other sciences, because there is very little logical connection between mathematical induction and induction in the other sciences.

The principle of mathematical induction may be stated as follows: Let $S(n)$ be a statement involving a variable n which becomes a sentence (true or false) whenever a natural number is put in place of n. If $S(1)$ is true, and whenever $S(n)$ is true $S(n+1)$ is also true, then $S(n)$ is true for every natural number n.

Many dangers are involved in making use of the inductive process (as distinguished from mathematical induction). Consider the function $F(n)=n^2-n+41$, each of the values of n, form 1 to 40 will produce a value for $F(n)$ which is a prime number. The one counterexample, $n=41$, however, destroys any conclusion one might be tempted to draw.

Induction can be used only to lay the foundation for deductive demonstrations. It cannot be used, or even thought of, as a proof.

CHAPTER III

THE C.E.E.B. COMMISSION ON MATHEMATICS

In the last ten years, several committees have been studying the secondary school mathematics program in North America and have published findings. Included among these are The School Mathematics Study Group which publishes a Newsletter from time to time on its work,[1] the Ball State Experimental Program[2] which has recently published texts with a modern approach,[3] the University of Illinois Committee on School Mathematics, set up in 1952 by mathematician Dr. Max Beberman, which is still working on a high school mathematics program being taught in at least ninety-five schools of the United States,[4] and the Commission on Mathematics appointed in 1955 by the Mathematics Examiners of the College

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[1]These may be obtained by writing School Mathematics Study Group, School of Education, Cedar Hall, Stanford University, Stanford, California.

[2]See Charles Brumfiel, Robert Eicholz, and Merrill Shanks, "The Ball State Experimental Program," The Mathematics Teacher, LIII, 2, (February, 1960).

[3]Brumfiel, Eicholz, and Shanks, Geometry (Reading, Massachusetts: Addison Wesley Publishing Company, Inc., 1960) and Brumfiel, Eicholz, and Shanks, Algebra I (Reading,Massachusetts: Addison Wesley Publishing Company, Inc., 1961).

[4]See Max Beberman, An Emerging Program of Secondary School Mathematics (Cambridge, Massachusetts: Harvard University Press, 1958).

Entrance Examination Board to study the mathematics curriculum of the high schools in the U. S. A.[5] In the present chapter the findings of this Commission will be studied at length.

The Commission came into being as a result of concern felt by the mathematics examiners of the College Entrance Examination Board about the curriculum they were testing. It was formed to consider and make recommendations towards the improvement, modernization and modification of the college preparatory courses of the secondary schools in the U. S. A.

One of the basic conclusions reached by the Commission is seen in its observation that mathematicians have changed in their approach to mathematics and the content of mathematics has changed.[6]

> The traditional secondary-school program in mathematics, made up of elementary algebra, plane geometry, intermediate algebra, solid geometry, trigonometry, and advanced algebra, consists almost entirely of mathematics developed over three hundred years ago, gradually introduced into the schools over the past one hundred and fifty years and crystallized into essentially its present form approximately sixty years ago. The subject matter was chosen and the presentation organized in accordance with an attitude toward mathematics that is

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[5] Albert E. Meder, Jr., "Proposals of the Commission on Mathematics of the College Entrance Examination Board," New Developments in Secondary-School Mathematics, Reprinted from the Bulletin of the National Association of Secondary-School Principals No. 247, May 1959 for the National Council of Teachers of Mathematics, 1201 Sixteenth Street, N. W., Washington 6, D. C., p. 19.

[6] Meder, p. 19.

> now antiquated and has been discarded by present day working mathematicians. The curriculum contains much obsolete material and, instead of being oriented to the needs of the second half of the twentieth century, is designed to meet the needs of science and technology as these needs existed some seventy five years ago.[7]

The Commission strongly recommended that the traditional requirements in elementary algebra, intermediate algebra, advanced algebra, plane and solid geometry, and trigonometry be replaced by a new formulation of college entrance requirements in mathematics. The new program would designate the new requirements in terms of length of time spent in study. The titles suggested for courses to be taught in four successive high school years were Elementary Mathematics I, Elementary Mathematics II, Intermediate Mathematics, and Advanced Mathematics. The Commission defined in rather specific detail the content of each year's program. In order to help implement its recommendations for revised and renewed emphasis on the content of instruction, the Commission published a separate volume of appendices concurrently with its report.

In one of its early publications, the Commission pointed out six specific areas in the present curriculum needing revision:

> (1) Too much attention is given, particularly in algebra, to routine manipulation in artificial situations, and not enough emphasis is laid on fundamental concepts.
> (2) Deductive reasoning is taught chiefly in connection with plane and solid geometry, and its application to other

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[7]Modernizing the Mathematics Curriculum, (New York: Commission on Mathematics of the College Entrance Examination Board, 1958), p.2.

parts of mathematics is largely ignored. Its use in algebra and trigonometry should be expanded.

(3) Too often the usual geometry course consists of rote memorization of sequences of theorems and fails to explain the deductive process clearly.

(4) Many topics which are now included were important at one time for applied science, but have become obsolete. These should be replaced by topics of current importance. Examples of obsolete topics are extensive solution of triangles by logarithms, deductive methods in solid geometry, and Horner's Method for finding roots of a polynomial.

(5) Many newer topics of importance in mathematics and its applications have little or no place in the course of study. Examples of modern subjects which might be included are descriptive statistics, statistical inference, elementary property of sets, and the basic ideas of modern algebra. Many of these topics are more elementary than topics now in our secondary school curriculum.

(6) Mathematics is too often presented as a series of isolated tricks, so that students get no view of the subject as a whole, and do not realize its position as a creative endeavor in our civilization.[8]

The Commission was completely convinced that the following three principles are fundamental to the formulation of a high school mathematics curriculum orientated to the needs of the present and the future:

(1) The proposals must be based on the existing curriculum, and must consist of modification, modernization, and improvement of the present pattern, rather than its discontinuance and replacement by entirely new content.

(2) The point of view of modern mathematics . . . must be used as a guide in determining the modifications to be made.

(3) Changes to be proposed must be sufficiently far reaching so that the modified curriculum is truly orientated to the present and future needs, but not so radical as to be beyond the competence of the available teaching staff. However, a willingness upon the part of school officials to participate

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[8] <u>Objectives of the Commission on Mathematics of the College</u> Entrance Examination Board, (New York, 1957), pp. 6-7.

in programs of in-service education must be assumed.[9]

The Commission was equally convinced of the need for revision in the branches of mathematics presently taught in secondary schools and its views warrant examination in each of the three branches. Let us begin with algebra.

If one were asked to give an example of rigorous mathematical reasoning, the most probable answer would be geometry rather than algebra. Euclid's Elements have stood for two thousand years as the supreme illustration of the mathematical manner of reasoning. Axiom, theorem, corollary; the defence of every statement by reference to a previously established truth -- this method of mathematical reasoning calls up in our minds a textbook in geometry, never a textbook in algebra.

Indeed until recently, elementary algebra has been largely a miscellaneous collection of rules for the manipulation of algebraic expressions, and it is not the developed science that elementary geometry has been for many years. In fact, if it were not for the study of plane geometry in our schools, it is doubtful whether from their study of algebra alone, our students would ever derive any clear notion of what is meant by the mathematical method.

This fact is more remarkable because algebra, dealing with concepts of a simpler nature, is better suited than geometry to serve

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[9] Modernizing the Mathematics Curriculum, p. 7.

as an illustration of what is essentially involved in mathematical reasoning. In geometry the very concreteness and familiarity of the subject-matter is apt to obscure the logical structure of the science, while in algebra the more abstract character of the content of the theorems makes it easier to fix attention on their formal logical relations. The Commission recognized the introduction of structure as a necessary revision in algebra and, " . . . while not discounting the manipulative skills necessary for efficient mathematical thought, puts chief emphasis on the structure or pattern of the system and on deductive thinking."[10]

Turning to geometry the Commission noted that, while it is a common belief that plane geometry has remained unchanged since written by Euclid over two thousand years ago, this belief is really not true. The Elements contain many logical gaps, and means to remedy these have been known for about sixty years. Indeed, it is not surprising that such an early and extensive application of the mathematical method as that of Euclid contains gaps.[11] The greatest of these defects are tacit assumptions used in the proofs but not found in the earlier work.

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[10] Program for College Preparatory Mathematics, Report of the Commission on Mathematics, (New York: College Entrance Examination Board, 1959), p. 2.

[11] Supra, pp. 10-11.

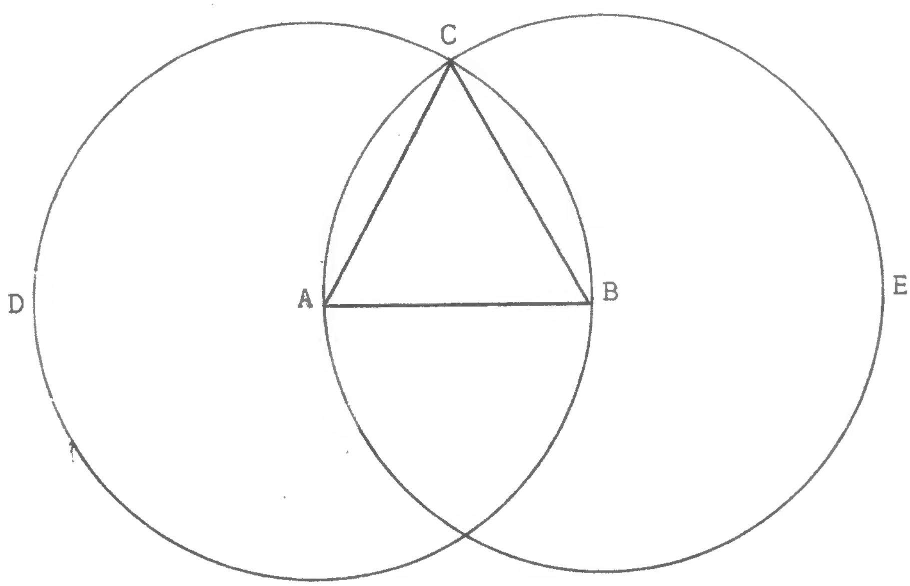

Fig.4.--Diagram for Book I Proposition 1

An example of a tacit assumption by Euclid not contained in the first principles is the first propostion of the Elements. An examination of Proposition I of Book One will show what is meant.

On a Given Finite Straight Line to Construct an Equilateral Triangle

Let AB be the given finite straight line.

Thus it is required to construct an equilateral triangle on the straight line AB.

With centre A and distance AB let the circle BCD be described; [Post. 3]
again, with centre B and distance BA let the circle ACE be described; [Post. 3]
and from the point C, in which the circles cut one another, to the points A, B let the straight lines CA, CB be joined. [Post. 1]

Now, since the point A is the centre of the circle CDB, AC is equal to AB. [Def. 15]

Again, since the point B is the centre of the circle CAE, BC is equal to BA. [Def. 15]

But CA was also proved equal to AB; therefore each of the straight lines CA, CB is equal to AB.

And things which are equal to the same thing are also equal to one another; therefore CA is also equal to CB. [C. N. 1]

Therefore the three straight lines CA, AB, BC are equal to one another.

Therefore the triangle ABC is equilateral; and it has been constructed on the given straight line AB.

(Being) what it was required to do.[12]

There is a difficulty in the above proof and it is not, as might first be supposed, that the two circles intersect at two points. Rather the flaw is in the assumption that the circles

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[12] The Thirteen Books of Euclid's Elements, Great Books of the Western World, ed. Robert Maynard Hutchins, 11, (Toronto: Encyclopaedia Britannica, Inc., 1952), pp. 2-3.

intersect at all. This is not evident, nor does it always happen. Two circles in the same plane can lie entirely outside each other, or one may lie entirely inside the other. One might say that looking at them will show that they must intersect. This is what Euclid did and that evidence is not admissible in this court, for there is not any proposition which states this.[13] This is one of the types of incompleteness in Euclidean geometry which clearly shows need for revision. However, only now are texts being printed which present plane geometry from an elementary approach and which are mathematically adequate.[14]

The Commission found the same need for revision in trigonometry. The most numerous and important applications of trigonometry to mathematics and to modern science and engineering are found not in the solution of triangles but rather in vectors and their components. Unfortunately, the material taught in many high schools gives undue emphasis to the solution of triangles which is made obsolete by modern methods of computation employing calculating machines rather than logarithms. Many students think that the prime

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[13]For a further discussion see Howard Eves and Carroll V. Newsom, <u>An Introduction to the Foundations and Fundamental Concepts of Mathematics</u> (New York: Rinehart and Company, Inc., 1958), pp. 37-41. Albert E. Meder, Jr., What is Wrong with Euclid?," <u>The Mathematics Teacher</u>, LI, 8 (December, 1958), pp. 578-584. <u>Appendices</u>, Report of the Commission on Mathematics, (New York: College Entrance Examination Board, 1959), pp. 109-111 and 166-174.

[14]Brumfiel, Eicholz, and Merrill, <u>Geometry</u>, p. ix.

objective of a course in trigonometry is the solution of triangles.[15]

A careful criticism of the present status of trigonometry is offered by Richard V. Andree:

> Twenty-two hundred years ago, Erathosthenes calculated the earth's radius, correct to within twenty-five miles, by measuring the angle that the sun's rays made with a vertical rod at Alexandria and at Aswan. At this time the most important aspect of trigonometry was the computation of the sides and angles of triangles. Today, engineering still demands this numerical skill, but other phases of trigonometry have become so important that the numerical solution of triangles is only a minor section in a course in modern trigonometry. Some schools still spend a third or more of their time on logarithms and the solution of triangles. This is not in the best interests of students who plan to enter engineering, science, or mathematics, and is certainly of little use to the student seeking a general education. Today's civilization demands a careful study of functional graphs (harmonics) and the dextrous use of identities, including the half-angle and double angle forms. The solution of trigonometric equations and identities is vital in modern science.
>
> The engineer of today is busy with problems that were unthought of twenty years ago. As a result of this, many shop technicians are doing work which was done by the engineer twenty years ago. The technician who is to get ahead today must be mathematically prepared to accept some of the responsibility which previously rested on the shoulders of the engineer. The engineer must be prepared to reach new horizons. . . .
>
> After interviewing industrialists, scientists, engineers, technicians, and shop mechanics, my conclusion is that the trigonometry which is most needed by modern civilization is:
>
> (1) The graphs of trigonometric functions. The student must realize in graphing y=2x+7Cos5x that x need not be thought of as an angle.
>
> (2) Identities. Students should be given only one member of an identity such as (1 + Cos 2x) (Csc 2x - Cot 2x) to simplify as much as possible, without realizing that their goal is Sin 2x.
>
> (3) Trigonometry equations, especially of higher degree.
>
> (4) The inverse trigonometric functions.
>
> (5) Definitions and applications involving polar coordinates, De Moivre's theorem, and complex numbers.

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[15] Cf., Modernizing the Mathematics Curriculum, p. 11.

(6) Practical problems involving equations and identities.

(7) Electrical engineers and technicians need the relationships Sin $\theta = (e^{i\theta} - e^{-i\theta}) / 2i$ and cos $\theta = (e^{i\theta} + e^{i\theta}) / 2$ and their ramifications along with De Moivre's theorem.

If your school is still teaching trigonometry of 250 B.C., then perhaps you should learn more about the trigonometric demands of modern civilization. Your school may use one of the several excellent texts on modern trigonometry and still be giving a 250 B. C. course by slighting these important modern phases. I hope not, because much of the difficulty students encounter in calculus, mechanics, and industrial applications is traceable directly to the lack of dexterity in the manipulation of trigonometric equations and identities.[16]

Because of the general need for revising the mathematics program, the Commission on Mathematics was asked also to recommend and suggest changes in the college preparatory mathematics curriculum. It is therefore fitting that we should know the premises on which the recommendations for the new curriculum are based.[17]

Although the Commission realized that the secondary school must serve all its students, it does not subscribe to the view that the entire school population should take exactly the same course. It believed that all secondary students need mathematics, but the Commission recommended all students need not take the recommended curriculum. It is designed for the students who can benefit from it.

The Commission realized that secondary schools must serve the

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[16]Richard V. Andree, "Modern Trigonometry," The Mathematics Teacher, XLVIII, 2 (February,1955), pp. 82-83.

[17]See Program for College Preparatory Mathematics, pp. 10-16.

needs of those students not going to college, and indeed many aspects of the Commission's program can be adapted to a program for general education for such students. However, the program as set forth by the Commission was designed to meet the needs of the "college-capable", those capable of college work.

The Commission reported that it is of prime importance to the school, as well as the home, to see that students with college potential tackle this program for at least three years. The Commission continued, stating its belief that parents, teachers and counselors have a duty to see that as many as possible of the college-capable study high school mathematics for four years. The top students should attempt the Advanced Placement Program of the College Entrance Examination Board which offers these students college-level courses in their senior year.[18]

The Commission believed there are cogent reasons for recommending that college-capable students should study mathematics for four years. Many high school students do not know what career they would like to follow, and without mathematics many careers are eliminated. Further, mathematics is more easily understood by the young, as experience proves that most of our great mathematicians and scientists became interested in their field in high school.

Another finding of the Commission was that students of college preparatory mathematics should be taught in groups with similar

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[18] _Ibid._, p. 15.

interests and similar intellectual abilities. This type of instruction, it is contended, increases the challenge to the student and the likehood of his developing his talents and ability to the maximum.

The Commission did not recommend "practical" courses such as consumer mathematics, instalment buying, principles of insurance to the college-capable students. It is the belief of the Commission that these students have sufficient mathematical ability to acquire this information as the need for it arises without specific classroom instruction.

Another conclusion of the Commission was that it is the responsibility of mathematicians and mathematics teachers to decide what subject-matter is important, what should be taught and what is obsolete. The philosophers of education and psychologists, on the other hand, should decide the aims, learning theory, grade placement of material, methods and so on.

The Commission was of the opinion that calculus is a subject best left to the colleges, except for its inclusion in the Advanced Placement Program. Students, however, should be given a complete pre-calculus program.

The Commission stated that if curricular revision is to take place in mathematics, then mathematics teachers must take a lead in the revision. They must be aware of new developments and materials and with assistance can do much to improve the curriculum.

As well as its general suggestions for improvement, the Commission made specific recommendations for each branch of mathematics. In order to improve the study of algebra the Commission recommended[19] that more attention be given to algebra as a part of the secondary school curriculum and that it be presented from a contemporary point of view. The teaching of algebra, the Commission asserted, should not consist primarily in the teaching of manipulative skills. Although skills are necessary, it is much more important that students have a good understanding of the deductive reasoning involved. The teaching of algebra should be directed toward the development and understanding of a number field. Too often algebra is taught as a set of "rules". The Commission advocated establishing the "laws" of algebra (axioms of a number field) because algebra is then more easily understood and becomes more meaningful.

Algebra has been largely transformed by mathematical research during the past quarter-century. The axiomatic development of algebras has brought new emphasis on the fundamental ideas and concepts of the subject. For this reason the Commission stressed emphasis on the nature of number systems and the laws for addition and multiplication (commutative, associative and distributive), the meaning of conditional equations, identities and inequalities.

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[19] *Ibid.*, pp. 20-22.

The nature of a function--particularly the linear, quadratic, exponential, and logarithmetic functions--should be included in the curriculum.

If the above mentioned laws are thoroughly understood as recommended by the Commission, then the common "rules" for removing parentheses, factoring, multiplying polynomials and the manipulation of fractions are no longer necessary. Therefore, algebra should be presented as a deductive system with its definitions and axioms. This form of presentation will enable the student to understand more fully the nature of the subject and will increase his ability to solve more problems. This in turn will avoid solutions by rote methods or solutions by types.

The Commission stated that the teaching of high school geometry has three main objectives:[20] 1) to acquire information about geometric figures both in the plane and in space. Insofar as geometry is a model of the physical world, every student needs the facts of geometry to deal with the physical world, and further geometry is a prerequisite to trigonometry and calculus; 2) to develop an understanding of the deductive method as a way of thinking. It has already been pointed out that it is desirable to use the deductive method in all mathematical subjects, therefore the time devoted to it in geometry can be reduced; 3) to provide opportunities for original and creative thinking by students.

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[20] *Ibid.*, pp. 20-28.

Since geometry can be easily grasped by all students and is a challenge to all, no matter how intelligent, a large part of the course can be devoted to the solution of original exercises, and to discovering and proving relations.

Since recent developments in geometric thinking have disclosed grave faults in the logical structure of Euclid,[21] the traditional approach to high school geometry must be modified.

Since both algebra and geometry are mathematical models, consideration must be given to this important system of mathematics. A mathematical model consists of a set of undefined terms (basic definitions) and unproved propositions (axioms). All other concepts are defined in terms of these and all other propositions are proved by means of these. Further, the undefined and unproved elements should be as few as possible.

The Commission stated that for high school purposes it is not necessary that these unproved propositions be as few as possible, so long as they are consistent. This will permit the number of proved theorems to be reduced and thus remove certain difficulties. It will also permit the student to spend more time on original exercises, time previously devoted to learning theorems.

Coordinate geometry should be introduced early in the

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[21]See <u>supra</u>, footnote 13, p. 52.

course, preferably after the first sequence of theorems, asserted the Commission. The student will then be able to combine geometric facts and graphical algebra. Furthermore, he will receive a good beginning in work he will need later, for mathematics from calculus on employs geometric material in analytic form.

Whitehead points out the importance of coordinate geometry in his Introduction to Mathematics.

> No one can have studied even the elements of elementary geometry without feeling the lack of some guiding method. Every proposition has to be proved by a fresh display of ingenuity; and a science for which this is true lacks the great requisite of scientific thought, namely, method. Now the essential point of coordinate geometry is that for the first time it introduced method. . . . [It] relates together geometry, which started as the science of space, and algebra, which has its origin in the science of number.[22]

Further, the geometry course should not be divided into parts on the basis of dimensionality. It is desirable, the Commission believed, to teach plane and solid geometry together. For example, the sphere can be considered along with the circle. The object of solid geometry is to teach spatial relations and spatial perception. Theorems in spherical geometry are good material insofar as they can be contrasted with plane geometry.

The material of solid geometry should include the basic facts about lines, planes, angles, dihedral angles, and spheres,

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[22]Alfred North Whitehead, Introduction to Mathematics (New York: Oxford University Press, 1958), pp. 83-84.

according to the Commission. These need not be established deductively, although a clear understanding of the relationships is necessary.

Students should also be aware that there are other "non-Euclidean" geometries, stated the Commission. Although they need not study any of these, they should know that they are "true" in their own field. They should know, for example, that on a sphere (Riemannian geometry) the sum of the three angles of a triangle always exceeds two right angles.

The Commission recommended a complete revision and reorganization of the trigonometry course. Trigonometry is the branch of secondary mathematics most clearly related to technical applications. In the past these were mainly concerned with navigation and surveying. Now the fields of statics and dynamics, electromagnetic waves, and vibration problems present challenging applications.

Therefore, according to the Commission, the trigonometry course must be reorganized. Further, the stress previously given to the solution of triangles and identities must be replaced by attention to vectors and functional properties. This does not mean that the solution of triangles is no longer important. Rather, it accepts the fact that new means of handling them have appeared, namely computing machines and special tables.

The Commission recommended that the following units be incorporated into the secondary school trigonometry course: (1) Rudimentary trigonometry of right angles. (2) Trigonometry of x, y, r, θ, --- coordinates, vectors, complex numbers. (3) Cosine and Sine laws, addition theorems, identities. (4) Circular measure, circular functions and their wave nature.[23]

The following nine-point program for college-capable students in the Commission's view will summarize appropriately what has been said.

(1) Strong preparation, both in concepts and in skills, for college mathematics at the level of calculus and analytic geometry
(2) Understanding of the nature and role of deductive reasoning-in algebra, as well as in geometry
(3) Applications of mathematical structure ("patterns") -for example, properties of natural, rational, real, and complex numbers
(4) Judicious use of unifying ideas-sets, variables, functions, and relations
(5) Treatment of inequalities along with equations
(6) Incorporation with plane geometry of some coordinate geometry, and essentials of solid geometry and space perception
(7) Introduction in grade 11 of fundamental trigonometry-centered on coordinates, vectors, and complex numbers
(8) Emphasis in grade 12 on elementary functions (polynomial, exponential, circular)
(9) Recommendation of additional alternative units for grade 12: either introductory probability with statistical applications or an introduction to modern algebra.[24]

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[23] _Program for College Preparatory Mathematics_, pp. 28-30.

[24] _Ibid._, pp. 33-34.

CHAPTER IV

RECOMMENDATIONS FOR THE NOVA SCOTIA CURRICULUM

Every student, whatever his capabilities should be given opportunity and encouragement to develop his talents. It must be conceded, of course, that administrative provisions for the pupil of mathematics depend largely on the size of the school. In the small high school, where ability grouping is not practical, enrichment procedures must be developed by the mathematics staff to meet the unique needs of the academically talented pupils.

Recommendation 1 Students should be grouped in mathematics classes according to ability in mathematics.

This type of grouping is not the same as across-the-board grouping where the student is grouped according to general ability.[1]

Dr. Harry D. Lead made the following comments on ability grouping at a seminar held by the Canadian Teachers' Federation:

> It seems that the tremendous growth in school population has caused an even greater range of abilities in mathematics and, as a result, our teaching, which is directed at the average pupil, is missing both the slow learner and the very capable student. Homogeneous grouping can permit teaching at several levels--the more levels the better! Some schools could have slow, low average, high average and advanced groups. Homogeneous grouping along with subject promotion permits the grouping by subject without

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[1]See James Bryant Conant, The American High School Today (Toronto: McGraw-Hill Book Company, Inc., 1959), p. 49.

> the stigma of being a "bobo"! The home room has a miscellaneous heterogeneous group but they subdivide for individual subjects. In this way many students in the advanced maths group might be in the below-average History or English group, or vice-versa, so the problems of individual differences are taken care of within each subject.[2]

Ability grouping, however, should not be considered an end in itself. It becomes justified only as it makes for a better curriculum and more productive methods of teaching for all pupils.

There are several recommendations that apply to the teacher of mathematics, the crucial element in providing a suitable program for the college-capable student. Ideally, the teacher of mathematics, besides possessing the characteristics of a good teacher, must have a rich and recently refreshed background in mathematics. There are schools fortunate enough to have such staff members; and they will, therefore, have no serious problems in initiating, maintaining, and constantly improving their programs.

In other schools, too, there will be teachers who are enthusiastically willing to undertake such a program. They should be encouraged to do so, but provisions must be made to make it possible for them to prepare themselves for the job.

In still other schools, there may be need, first of all,

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[2] Dr. Harry D. Lead, "Who Should Study Mathematics--And How Much?," New Thinking in School Mathematics, Report of a seminar held by the Canadian Teachers' Federation at Ottawa, April 28-30, 1960, p. 144.

for laying a foundation for the introduction of the program. To this end the following recommendations are made:

Recommendation 2 Teachers of mathematics should learn their mathematics from the mathematicians, their methodology from the faculty of education, and should get further "on-the-job" training by talking things over with other classroom teachers.

This point is made by R. E. K.Rourke, an Executive Director of the Commission on Mathematics, in the following words:

> Teacher education in mathematics should have three objectives: first, to develop in the teacher a valid and effective understanding of the nature of mathematics; second, to familiarize him adequately with the subject matter of the field, particularly with those parts he will be called upon to teach; and finally, to equip him with a sound and effective methodology and teaching techniques. In other words, teacher education must deal with philosophy, subject-matter and methodology.
>
> Not the least important of these is philosophy. One of the major shortcomings of mathematical instruction at present is the inadequate understanding of the nature of mathematics held by many teachers.
>
> The popular conception of mathematics, probably shared by most teachers, is well epitomized by the catch-phrase, "Two and two are four," proverbially cited as an example of a typical mathematical truth. But in fact this phrase is an extremely poor illustration of a mathematical truth, and reflects a mechanical, computational view of the nature of mathematics that is both erroneous and harmful.
>
> The essence of mathematics is not computation, and the nature of mathematics is anything but mechanical. Mathematicians do not spend their time grinding out specific facts nor routine deductions. Imagination, abstraction, and generalization are characteristic of the nature of mathematics; as a philosopher of a bygone generation once observed, deductive reasoning is but the pavement on which the chariot of the mathematician rolls.

. . . .

If mathematics on any level is cenceived of as a purely mechanical skill, like typewriting, that can be learned by practice and drill, and can be transmitted by anyone who knows it to a neophyte by showing him the procedure and drilling him, no effective program of teacher education can be constructed.

Too often the nature of arithmetic has been so conceived; the multiplication tables are to be learned by rote, just as the proper finger for each letter on a typewriter is to be noted and its use made habitual by meaningless drills such as asdfg hjkl;. (This analogy may be unfair to the study of typewriting, for even here relationships are emphasized in a limited way.) Anyone who has learned to compute can,with proper methodology, teach others to compute; no background of understanding is required.

Similarly, algebra has been thought of as a collection of computational tricks or manipulative devices, with the result that the outcome of instruction in algebra has merely been to enable the pupil to solve problems that resemble sufficiently closely the typical examples solved in the textbook or the classroom. Even geometry has been reduced, in too many instances, to rote memorization of theroms and their proofs.

The first requisite for an adequate and effective program of teacher education is the complete abandonment of such views with respect to any level of education whatever, from kindergarten to graduate school, A teacher of mathematics must understand the nature of the subject whether he is teaching "counting numbers" to first-grade pupils or linear algebras to graduate students. Mathematics is not the study and memorization of computational or other tricks once for all worked out in the past and transmitted from generation to generation. It is the imaginative, creative study of "pattern," of regularity in phenomena that can be recognized by the human mind; it involves abstraction and generalization. One must understand the nature of the subject before he can teach it. This is the first great task of teacher education in mathematics.

The second is easier: to give the prospective teacher knowledge of subject matter. Here we must only agree how much background knowledge is necessary in order that one may teach effectively at any specified level, and then provide the necessary time and appropriate courses in which this knowledge may be acquired.

Finally, adequate and effective teacher education must deal with the problem of the learner: how pupils grow and develop; how they (or adults) learn; how to select appropriate material for appropriate objectives; how to present it; how to evaluate what has been done; and the like. No one can teach effectively if he does not understand the psychological nature of learning nor how the objectives he seeks may be attained.[3]

Recommendation 3 The Canadian Mathematical Congress summer school should be strengthened by more financial support to it and to the teachers attending, this support to come from industry, the Department of Education and the school boards.

Recommendation 4 The Department of Education should issue specialist licences to teachers who have reached a determined proficiency in a particular subject, and school boards should be encouraged to have their staff teach their specialty. This point is stressed by Howard F. Fehr:

In fact, the teacher of mathematics should teach as B. V. Gredenko recently said the USSR demands that its teachers do.

"A teacher who reduces his task to the point that he only communicates to the pupil the sum of knowledge specified in the curriculum, and merely teaches the pupil to deal with routine problems, rarely achieves any success. From the teacher is demanded enthusiasm for his subject and the conviction that his subject is one of the most important affairs of the nation. From the teacher is demanded that he implant in the students a love for mathematics and a conviction of their creative powers in the subject; that he describe, in general outline before their intellectual gaze, the impressive picture of the

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[3] R. E. K. Rourke, "The Commission on Mathematics of the CEEB and Teacher Education," New Developments in Secondary-School Mathematics, pp. 173-175.

> uninterrupted development of mathematics with its limitless connections with technology, the natural sciences, and all the other manifestations of human activity."
>
> We cannot demand this from our teachers, but we can hope they will achieve a similar point of view. We need quality teachers.[4]

Further, every effort must be made to improve articulation between elementary and secondary programs of instruction in mathematics. This can be stated as

Recommendation 5 Fundamental concepts and principles of mathematics must be more thoroughly taught in the elementary school as well as in the secondary school.

Some of these concepts pervading elementary and secondary school mathematics are described in detail by H. Van Engen:

> Enumeration Systems. One of the first things a child learns in school and in the home is how to enumerate a group of objects. At first he does this by assigning individual names to collections; that is, one, two, three, etc. The child is soon taught that it is best to adopt a base (ten) for his enumeration scheme in order to simplify the system.
>
> As he progresses in his learning, he finds out that a base other than ten can be used to systemize counting. This opens the way for an extensive study of enumeration systems in the elementary and secondary school. Such a study would include the binary system which has assumed an ever greater importance in the past decade due to the extensive development of digital computers.
>
> The Number System. Once the child learns that the symbol '3' can be applied to a particular group of apples, grapes, dogs, planes, and so on ad infinitum, he has a

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[4]Howard F. Fehr, "The Mathematics Teacher, Present and Future," New Developments in Secondary-School Mathematics, pp. 169-170.

foundation for a major abstraction; namely, that '3' is attached to that which is common to all the above mentioned groups. Of course, this idea is never verbalized in the lower grades. While the germ of the idea is dominant at this stage, later it can be "removed" from the object for its full abstract value. Of course, the same process occurs for all numbers.

Soon after his first experiences with arithmetic operations, the pupil should be introduced to those properties which are common to all those things we call numbers. Let us assume that the pupil is studying whole numbers (positive) and has not, as yet, thought of fractions as numbers. At this stage he should learn that:

1. When we add or multiply any two whole numbers, we always get another whole number. (This doesn't happen for subtraction and division. Thus $16 \div 5$ is not a whole number.)

2. The numbers are interchangeable for addition and multiplication. For example, $2 \times 3 = 3 \times 2$ and $3 + 2 = 2 + 3$.

3. Grouping numbers by twos in any order is always permissible in multiplication and addition. For example, $3 \times (2 \times 4) = (3 \times 2) \times 4$ and $(3 + 2) + 4 = 3 + (2 + 4)$.

4. The operation of multiplication may be "distributed" over the operation of addition, for example, $2(6 + 4) = 2 \times 6 + 2 \times 4$.

Having learned these fundamental principles in the elementary school, the pupil learns to develop a general way by which to state these principles in the secondary school and he learns that any collection of mathematical entities which possess the properties listed above will be called numbers. Thus he finds that fractions, positive numbers, negative numbers, and complex numbers all possess these properties. Of course, they possess others as well, but such things as $3 + i$ and $2 - 5i$ are numbers because they can be manipulated in the same way the whole numbers were manipulated in the above examples.

Number Pairs and Sets of Number Pairs. The elementary school pupil soon learns that it takes two or more numbers used as pairs to describe some physical situations. He learns to use the form a/b as a pair of whole numbers to describe situations involving fraction ideas, and he learns to use this same form to describe ratio situations.

Late in the elementary school, the pupil works with sets of number pairs. Thus he might make up a table of number pairs showing how much 1, 2, 3, 4, . . . 12 tickets to the school circus will cost at 25 cents each. This is a number-pairing activity. A number representing the number of tickets purchased is paired with the number representing the cost of the tickets. In an arithmetic class, the pairs may be graphed to obtain a visual way of reading the pairs or to see the general trend of increased cost versus increased number of tickets purchased.

In the secondary school, pupils learn to write a rule ($C = 25n$) showing how the numbers are to be paired. Such activities serve as an introduction to the idea of variable, formula, set of numbers and a set of number pairs, and generally to the subject of algebra itself. Advanced work will introduce such topics as relations, functions, and set-functions, all of which are based on the pairing of numbers, writing rules to enable one to pair numbers, and similar activities.

Geometric Concepts. The study of geometrical forms and their properties is an integral part of mathematics. As with number ideas, certain very important geometric ideas have their origins in the elementary schools.

. . . .

These ideas must be emphasized in elementary instruction. Too frequently elementary (and secondary) instruction ignores fundamental concepts and places its emphasis on memorization of facts without providing the cement to make the facts bind into a whole. This is why some feel that the elementary school stunts the mathematical growth of children. Children can grasp abstract ideas; in fact, they enjoy them. The forward movement in mathematics can only come if we learn how to teach abstractions and how far pupils of varying abilities can go in making mathematical abstractions.[5]

The demand for scientists and engineers, all of whom must have a sound knowledge and understanding of mathematics, is growing. New applications of mathematics in industry and in other branches of economic activity are leading to a demand for more mathematicians

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[5] H. Van Engen, "Concepts Pervading Elementary and Secondary Mathematics," New Developments in Secondary-School Mathematics, pp. 116-118.

with new kinds of skills. All these demands are creating a need for a re-appraisal of the content and methods of school mathematics.

Despite the great amount of discussion and study of the problems of mathematics teaching, much of it is not having the desired impact on the schools. In fact in Canada, as stated in New Thinking in School Mathematics (a report of the Organization for European Economic Co-Operation), "there is no trend towards change. Present programme was established in 1900. No national movement to re-examine the programme. [sic] Several committees are studying possible reforms for the programme."[6] This report goes on to say that Canada and the United States "are the only two countries [of the twenty reporting] which have no distinct mathematics programmes for college-preparatory students of varying interests."[7]

In the light of these facts the following recommendations are made:

Recommendation 6 A committee consisting of mathematicians (university professors or other professional mathematicians) and teachers of mathematics should be established to examine the Nova Scotia mathematics curriculum with a view to its revision in the light of new developments.

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[6] New Thinking in School Mathematics, Organisation for European Economic Co-Operation, Office for Scientific and Technical Personnel, p. 181.

[7] Ibid., p. 194.

Recommendation 7 A summer school should be established to offer courses not only for those students who need remedial work in mathematics, but also for bright and ambitious students who wish to use the summer to broaden their horizons in mathematics.

Recommendation 8 Extra-curricular activities for above average students of mathematics should be encouraged. These activities might include such things as mathematics clubs, mathematical contests, mathematics field days, exhibits and fairs.

Recommendation 9 The provincial examination system should be reviewed in the light of the following remarks on the defects of centralized examinations:

> Though a centralised system of examination has the advantage of maintaining a fixed common standard in a definite body of knowledge, it also has disadvantages. In the first place, most countries develop a certain pattern of examination questions, and teachers spend much time training their pupils how to answer these examinations-time which could better be spent on teaching more mathematics. Many of these examinations become intelligence and endurance tests, rather than tests of mathematical reasoning and comprehension.
>
> Furthermore, it is exceedingly difficult to get a central educational agency to adapt its examinations to experimental developments in the teaching of mathematics. It is difficult to set an examination when two or more radically different approaches to a subject must be covered-for example, two completely different axiomatic treatments of plane geometry. Examinations tend to standardise teaching, thus reducing teachers' opportunity or desire to show originality in classroom work.[8]

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[8] *Ibid.*, pp. 97-98.

Conclusion

Whatever the results of the mathematics' curriculum revision due to the new developments may be, it is safe to say that some of the immediate effects will be a greatly increased emphasis on structure, a reconsideration and refinement of definitions, a clearer attention to the deductive process, a better balance in the treatment of equations and inequalities, a broadening of the concepts of algebra and geometry, a revised treatment of trigonometric functions and techniques, a fundamental treatment of probability and statistical inference, and a reevaluation of the significance of many of the manipulative procedures at all levels of teaching.

Teachers must keep in mind that there is good and bad in both the old and the new in mathematics. The possible contributions of each to the mathematics programs of our schools must be weighed on the scales of effective and efficient training in mathematics in an era when technological patterns are subject to rapid and drastic change. Such change can outmode in rapid and serious fashion mere manipulative procedures, but the basic structure of mathematical systems is free from alteration or substitution. This significant fact very possibly may serve as the keystone to all future curriculum planning in both the elementary and secondary schools. If this should be the case, then the impact of modern mathematics on the curriculum of our secondary schools will be tremendous.

APPENDIX I

OUTLINE OF PRESENT NOVA SCOTIA MATHEMATICS CURRICULUM

Algebra

Grade IX Text: Petrie, et al.: Intermediate Mathematics Book Three.

I Algebraic notation

II Operations with signed numbers

III Simple equations

IV Special products

V Factoring (a) trinomials

(b) difference of two squares

VI Simplification of fractions

VII Multiplication and division of fractions

VIII Addition and subtraction of fractions

Grade X Text: Wells and Hart: Modern Second Course in Algebra

IX Fundamental operations

X Special products

XI Factoring

(a) Highest common factor

(b) Difference of two squares

(c) Difference of two cubes

(d) Sum of two cubes

(e) Trinomials

(f) Grouping

XII Simplification of fractions

XIII Multiplication and division of fractions

XIV Addition and subtraction of fractions

XV Fractional equations

XVI Problems

XVII Graphs of linear equations

XVIII Algebraic and graphical solutions for linear systems of equations

XIX Simplification of radicals

XX Simple radical equations

Grade XI Text: Wells and Hart: op. cit.

XXI Solutions of quadratic equations

(a) factoring

(b) completing the square

(c) formula

(d) graphical

XXII Fractional equations reducible to quadratics

XXIII Imaginary numbers

XXIV Graphs of

(a) circle with centre at origin

(b) ellipse with centre at origin

(c) parabola

XXV Algebraic and graphical solution of systems involving quadratics

XXVI Exponents and radicals

XXVII Radical equations

XXVIII Logarithms

XXIX Trigonometry of right triangle

XXX Arithmetic and geometric progressions

XXXI Ratio and Proportion

XXXII Variation

Grade XII Text: Petrie, et. al.: Algebra A Senior Course

XXXIII Factoring

XXXIV Surds and indices

XXXV Ratio and proportion

XXXVI Equations

XXXVII Progressions

XXXVIII Variation

XXXIX Functions

XL Permutations

XLI Binomial theorem with positive integral exponents

Geometry

Grade X Text: Oliver, Winters and Hodgkinson: *A First Course in Plane Geometry*.

I Axioms and definitions

II Congruent triangles

III Constructions

IV Parallel lines

V Parallelograms

Grade XI Text: Oliver, Winters and Hodgkinson: *op. cit.*

VI Areas of polygons

VII Loci

VIII Circles

IX Ratio and proportion

Trigonometry

Grade XII Text: Playne and Fawdry: Practical Trigonometry

I Solution of right triangles

II Area of triangles and pentagons

III Solution of triangles

IV Trigonometric equations

V Derivations and identities

VI Inscribed, escribed, and circumscribed circles

VII Graphs of trigonometric functions

VIII Distance between points on the surface of the earth

IX Distance and dip to the horizon

APPENDIX II

GLOSSARY OF TERMS

Analytic geometry: The essence of analytic geometry is that to every ordered pair of real numbers there corresponds a unique point in the plane, and that every point in the plane can be uniquely identified by an ordered pair of real numbers.

Analytic process: considers the desired conclusion and reasons that it is implied by the conclusion of some proposition.

*Associative: Addition is associative if $(a+b)+c=a(b+c)$. Multiplication is associative if $(a \times b) \times c = a \times (b \times c)$.

Axiom: A statement that is accepted as valid and in no case should be considered as a "self evident truth".

Binary operation: A binary operation on a set S is a correspondence which associates with each ordered pair of elements of S, a unique element which is also an element of the same set S.

Cartesian geometry: See analytic geometry.

Closure: A set is closed with respect to an operation if, when any two members of the set are combined by the operation, a member of the set is produced.

*Commutative: Addition is commutative if $a+b=b+a$. Multiplication is commutative if $a \times b = b \times a$.

Complement of a set A: A complement of a set A with respect to a universal set, U, is the set of all elements of U that are not contained in A. (Symbol A').

*Conjunction: $p \wedge q$. Translated as "p and q". (Symbol $\wedge$).

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*These terms are basic mathematical terms and as such are undefined. Only an explanation is attempted in this glossary.

Consistent: A set of axioms are said to be consistent if contradictory statements are not implied by the set.

Contrapositive: The contrapositive of $p \rightarrow q$ is $(-q) \rightarrow (-p)$.

Converse: The converse of $p \rightarrow q$ is $q \rightarrow p$.

Coordinate geometry: See analytic geometry.

Direct proof: A direct proof starts with the hypothesis and proceeds through a chain of syllogistic reasoning to the implication of the desired conclusion.

*Disjunctive: The disjunctive is $p \vee q$. This is translated as "p or q". (Symbol $\vee$).

*Distributive: Multiplication is distributive with respect to addition if $ax(b+c)=axb+axc$.

Empty set: A set that has no members. (Symbol $\emptyset$).

*Equivalence: $p \leftrightarrow q$. Translated as "p if and only if q". (Symbol $\leftrightarrow$).

Field: Any set of elements or any system of numbers which exhibits eleven special properties [see pp. 28-29. for these properties] is known as a <u>number field</u>, and the properties are called the <u>field properties</u>.

Finite geometry: A geometry, which, by the nature of the assumptions on which it is based, deals with a finite number of points and lines.

Group: A set of elements together with the specified rules of operating with those elements. [See pp. 24-25. for these rules].

*Implication: $p \rightarrow q$. Translated as "if p then q". (Symbol $\rightarrow$).

Indirect proof: An indirect proof is based on the fact that if $(-p)$ is false, then p is true. Hence to prove that p is true, we attempt to show that $(-p)$ is false.

Integral domain: An integral domain is a set of elements together with the specified rules of operating with these elements. [If we replace R9 of the set of field properties on pp. 28-29. by the cancellation property if axb=axc and a=0, then b=c the resulting system is an integral domain.]

Intersection of two sets: The intersection of the sets A and B is the set composed of those elements that are in both A and B. (Symbol $\cap$).

*Inverse: The inverse of $p \rightarrow q$ is $(-p) \rightarrow (-q)$.

Law of the syllogism: If the proposition p implies the proposition q and the proposition q implies the proposition r, then proposition p implies proposition r.

Mathematical induction: Let S(n) be a statement involving a variable n which becomes a sentence (true or false) whenever a natural number is put in place of n. If S(1) is true, and whenever S(n) is true S(n+1) is also true, then S(n) is true for every natural number.

Mathematical method: It considers mathematics as a collection of statements beginning with some unproved statements (axioms) involving some undefined terms (basic terms) in which all further statements follow logically from the axioms and all new terms are defined in terms of the undefined ones.

Modern mathematics: This is the axiomatic or postulational method which is characteristic of so much of the mathematical activity of this century.

Natural numbers: are the positive integers.

*Negation: (-p). Translated as "not p".

Non-Euclidean geometry: A non-Euclidean geometry is one which contradicts Euclid's parallel postulate.

Proper subset: If all the members of a set A are also members of a set B, A is called a subset of B. If B has members that are not in A, then A is a proper subset of B.

Ring: A ring is a not empty set, S, on which two binary operations are defined, and which has the following properties. (1) a+b=b+a; (2) (a+b)+c=a+(b+c); (3) For every element a in S there exists an element 0, such that a+0=a;

(4) If a is an element in S, then there is an element x in S such that $a+x=0$; (5) $(ab)c=a(bc)$; and (6) $a(b+c)=ab+ac$, $(b+c)a=ba+ca$.

*Set: Set is an undefined concept, but it can be thought of as a collection of objects.

Set of solutions: The set of solutions of an equation is the set of values which satisfy the variable of the equation.

Subset: If all the members of a set A are also members of a set B, A is called a subset of B.

Solution set of an equation: See set of solutions.

Synthetic process: This consists of drawing a series of necessary conclusions until the desired conclusion is reached.

Tautology: A proposition that is true regardless of whether its component propositions are true or false.

Union of two sets: The union of two sets A and B is the set that contains those and only those elements that belong either to A or to B (or to both). (Symbol $\cup$).

Unit set: A set of only one element.

Universal set: An overall set from which one or more subsets are chosen is an universal set. (Symbol U).

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