A Comparative Study of Altman's Z-score and A Factor Analysis Approaches to Bankruptcy Predictions

by

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Abstract

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This study uses an adapted factor analysis to recast Altman's Z-score model and compare the two approaches in terms of their prediction performance. First, a brief review of Altman's Z-score model and the model of factor analysis method is introduced. Then, some recent breakthroughs of factor analysis are presented to illustrate the theoretical benefits of adapting the method. The data used in this study are described and collected from annual reports of healthy companies and companies who applied Chapter 10K bankruptcies over the time period of 2003 to 2009. Using those data, this study adapts the factor analysis and obtains a new Z-score model. Through comparisons, this study finally evaluates both Altman's model and the new Z-score model. In conclusion, this study finds that in both aspects of coefficients of determination and predictabilities, the new Z-score model shows better performance than Altman's model, thus providing an updated and refined tool for bankruptcy prediction.

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Content

List of Tables

Chapter 1 Introduction

Altman's Z-score is a tool used widely in finance for analyzing and predicting the risks of bankruptcy of listed firms. The Z-score is a number computed as a sum of weighted of financial variables. The Altman's Z-score function is an equation resulted from computation of Multivariate Discrimination Analysis (MDA). The theory of MDA, which is used in computation of Z-score function, however, has some drawbacks. Those drawbacks can be improved by using the recent breakthroughs in Factor Analysis. Therefore, it is worthy using the new developments in factor analysis to estimate a new Z-score model for predicting bankruptcies of listed companies. In addition, Altman used companies' data from 1950s and 1960s in obtaining the original Z-score model, which might be outdated. Thus, using more recent data to estimate new Z-score model is essential to evaluate the ability of analyzing and predicting the risk of bankruptcies of listed firms. This study intends to apply the results from recent developments in factor analysis to the Altman's Z-score model and use updated data to conduct the parameters estimations. The purpose is to see if the new score model would outperform the original Z-score model.

Chapter 2.1 Review of Altman's Z-score Model

The widely popular Z-score function used for analyzing and predicting bankruptcies was first published in 1968 by Edward I. Altman (Altman, 1968). In Altman's study, the initial sample involved sixty-six corporations with thirty-three companies in each group in the time period of 1946 to 1965. The Z-score uses multiple inputs from corporate income statements and balance sheets to measure the financial status of a company. The inputs that Altman selected were from those financial reports that are one reporting period earlier than bankruptcies. The inputs that Altman used were twenty-two different financial ratios. Altman considered that these financial ratios were chosen to eliminate size effects. Those ratios were divided in five categories: liquidity, profitability, leverage, solvency, and activity. The reason for dividing the input variables in case 5 categories is ad-hoc. They are standard financial categories.

Altman applied linear multiple discriminant analysis (MDA) to find the best combination of five variables from an original set of variables. However, when applying the method of MDA, Altman could not avoid biased estimators. Altman himself admitted to the bias and tried the best way to minimize it. It is generally believed that the biased estimators come from two sources: sampling errors and searching (Frank etc., 1965). This is the first drawback of MDA – **the biased estimators.**

After computations, Altman obtained the Z-score model as following:

$$Z = 0.012X_1 + 0.014X_2 + 0.033X_3 + 0.006X_4 + 0.999X_5$$

Where,

$X_1 = {}^{net\ working\ capital}\!/_{total\ assets}$

$X_2 = {}^{cumulative\ retained\ earnings}\!/_{total\ assets}$

$X_3 = EBIT/Total\ assets$

$X_4 = market\ value\ of\ equity/book\ value\ of\ total\ liabilities$

$X_5 = sales/total\ assets$

After obtaining the parameters of the Z-score model, Altman conducted a test to assess the model's performance. The test was used to evaluate the prediction accuracy. He believed that the "measure of success of the MDA in classifying firms is analogous to the coefficient of determination (R^2) in regression analysis". The result of his test to the initial sample is shown in the following:

Table 1 – Altman's Test

	Number Correct	Per cent Correct	Per cent Error	n
Type I	31	94	6	33
Type II	32	97	3	33
Total	63	95	5	66

Source: Altman, 1968

Type I means Type I error, that is the probability of the error of the model mispredicts firms' failures in the set of existing firms. Type II means Type II error, which is the probability of the error of the model mispredicts firms' existing in the set of failure firms. The given of these potential test errors indicates that MDA analysis should be tested in order to obtain the model's coefficient of determination (R^2). Thus, the conduction of the test itself indicates another drawback of MDA **– the uncertainty of coefficient of determination level of the model after parameter estimations.**

In light of these two drawbacks of MDA, a factor analysis approach to obtain a new Z-score model for predicting the firms' bankruptcies is proposed in this study. The following section will introduce the classical factor analysis and show how the recent breakthroughs in factor analysis can be used in this study to improve the Z-score's predictability.

Chapter 2.2 The Classical Factor Analysis

Factor analysis was invented in 1904 by Charles Spearman. Factor analysis is one of the success stories of statistics in the social sciences. The reason for its wide appeal is that it provides a way to investigate latent variables, the fundamental traits and concepts in the study and evaluate of individual differences (Robert and Robert, 2007).

In this study, the method of factor analysis is used to construct a factor model of describing financial characteristics of companies and, then to provide a comprehensive score for each firm. These scores, will be used to compare with Altman's Z-score.

Suppose x is a random vector with p dimensions,

$$x = (x_1, \cdots, x_p)'$$

With the mean of

$$\mathrm{E}(x) = \mu = (\mu_1, \cdots, \mu_p)',$$

And the covariance matrix

$$\mathrm{Cov}(x) = \Sigma = (\sigma_{ij})_{p\times p}.$$

Suppose *x* is a factor analysis model with *k* factors such that *x* could be expressed as

$$x_1 = \mu_1 + \lambda_{11} f_1 + \lambda_{12} f_2 + \cdots + \lambda_{1k} f_k$$
$$\ldots$$
$$x_p = \mu_p + \lambda_{p1} f_1 + \lambda_{p2} f_2 + \cdots + \lambda_{pk} f_k$$

That is

$$x = \mu + \mathit{\Lambda} f + u \qquad \text{---- (1)}$$

Where:

$$\Lambda = (\lambda_{ij})_{p\times k} = (\lambda_{(1)}, \ldots, \lambda_{(k)}),$$

$$f = (f_1, \ldots, f_k)',$$

and

$$u = (u_1, \ldots, u_p)'$$

are unknown. Λ is the factor loading matrix, λ_{ij} is the factor loading, f is the common factor and u is *error* or, *specific factor*.

Often the following assumptions are made:

$E(f) = 0;$

$Cov(f) = I_k$, where I_k is the k×k identity matrix;

$E(u) = 0;$

$Cov(f, u) = 0;$

Factor analysis model usually requires that the error covariance matrix follow two constraints:

$Cov(u) = \Psi \geq 0$ (non-negative definite) ---- (2)

$$Cov(u) = \Psi = \begin{pmatrix} \psi_1^2 & 0 & \cdots & 0 \\ 0 & \psi_2^2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \psi_p^2 \end{pmatrix}, \text{ (diagonal)} \qquad \text{---- (3)}$$

Ψ is called the *errors covariance matrix* or, *specific covariance matrix.*

From the assumptions above, it implies a covariance structure

$$\Sigma = \Lambda\Lambda' + \Psi.$$

Also, from the constraints above

$q_j = \sum_{i=1}^{p} \lambda_{ij}^2$, ($j$ =1,…, k) (the sum of elements that in j^{th} column in Λ)

is defined as the sum of squares of the j^{th} column elements of Λ, called the *variance contribution*, which is a measure of common factor f_j explaining all the variables x; and

the sum of squares of the first k columns' elements of Λ:

$$\sum_{i=1}^{k} q_i \qquad \text{---- (4)}$$

is called the *cumulative variance contribution*, which is a measure of the common factors f_1,…, f_k explaining all the variable x (equivalent with the coefficient of determination and used in later, i.e. the R^2);

$$\sum_{i=1}^{k} q_i/p$$

is called the *proportion of cumulative variance contribution*.

In addition,

$$h_i^2 = \sum_{j=1}^{k} \lambda_{ij}^2 \text{ (i=1,...,p)}, h_i^2 \leq 1$$

i.e. sum of squares of the i^{th}-row elements in Λ, is called the *communality*.

Finally,

$$\sum_{j=1}^{k} q_j = \sum_{i=1}^{p} h_i^2 = \sum_{i=1}^{p} \sum_{j=1}^{k} \lambda_{ij}^2 = tr(\Lambda'\Lambda) = tr(\Lambda\Lambda')$$

where *tr* is the trace of a square matrix.

The task of factor analysis is to estimate

$$\Lambda = (\lambda_{ij})_{p\times k}$$

and

$$f = (f_1, \dots, f_k)'$$

and then give factor $\{ f_j \}$ a reasonable explanation.

Chapter 2.3 Advantages of Adapting Factor Model

In the previous section, I indicated some drawbacks of the MDA itself. In this section, I will explain in what ways factor analysis would have its advantages over MDA in study of the score function.

For the drawback of biased estimator of MDA, factor analysis (principal component method) can produce a factor model with unbiased factor score, i.e. the new Z-score. Therefore, I used the first theorem directly from He's study – **Theorem 1** (He, 2012).

Theorem 1: If $\Lambda=\Lambda^*$, then

i. The Thompson factor score of the regressor of Λ^* is the rotation of the first k standardized principal components, which are the linear combination of *x*. i.e.

$$f^* = \Gamma_k'\left(\lambda_1^{-1/2} t_1' x_1, \dots, \lambda_k^{-1/2} t_k' x_k\right) = \Gamma_k'(\lambda_1^{-1/2} t_1, \dots, \lambda_k^{-1/2} t_k)' x \quad \text{---- (5)}$$

ii. The regressed factor score f^* of the regressor of Λ^* is unbiased, and its average prediction error is smaller than other factor scores. Therefore, the regressed factor score of the loading matrix Λ regressed under the principal component is better.

The proof of **Theorem 1** is contained in **Appendix 1**. Basically, **Theorem 1** shows that by adapting factor analysis principal component method, we can get unbiased factor loading matrix, Λ^*. Then, after adapting the data in the factor-loading matrix, we can get unbiased factor scores, i.e. the Z-value in this case.

For the second drawback of MDA, i.e. uncertainty of coefficient of determination, factor analysis (principal component method) has its advantage. In factor analysis, $\sum_{i=1}^{k} q_i/p$ is a proportion of cumulative variance contribution. This proportion is analogous to the coefficient of determination in ordinary regression analysis, i.e. R^2. Thus, this proportion measures the percentage of data explained by factor model. Recent breakthrough in factor analysis (principal component method) can maximize the proportion. That is, when we adapt the principal component method of factor analysis, we can guarantee the factor model maximizes its ability to explain the data, i.e. maximize the coefficient of determination. In order to do so, I directly use the **Theorem 2** (He, 2012).

Theorem 2： For a given k＜r =rank(Σ),

(i) For any matrix of factor loadings, Λ, for (1) we have

$$\mathrm{tr}(\Lambda\Lambda') \leq \sum_{j=1}^{k} \lambda_j \qquad \text{---- (6)}$$

where *tr* is the trace of a square matrix.

(ii) If $\Lambda=\Lambda^*$, i.e. the matrix of factor loadings, estimated under the principal component method, then for (1) we have

$$max\left\{\sum_{i=1}^{k} q_i\right\} = tr[\Lambda^*(\Lambda^*)'] = \sum_{i=1}^{k} \lambda_i \qquad \text{---- (7)}$$

The proof of **Theorem 2** is contained in **Appendix 2**. **Theorem 2** indicates that, through rotation, factor model will find one loading matrix that maximizes the coefficient of determination of the factor model. Thus, no more need of using the modeling test to get the coefficient of determination.

Chapter 3 Sample Data

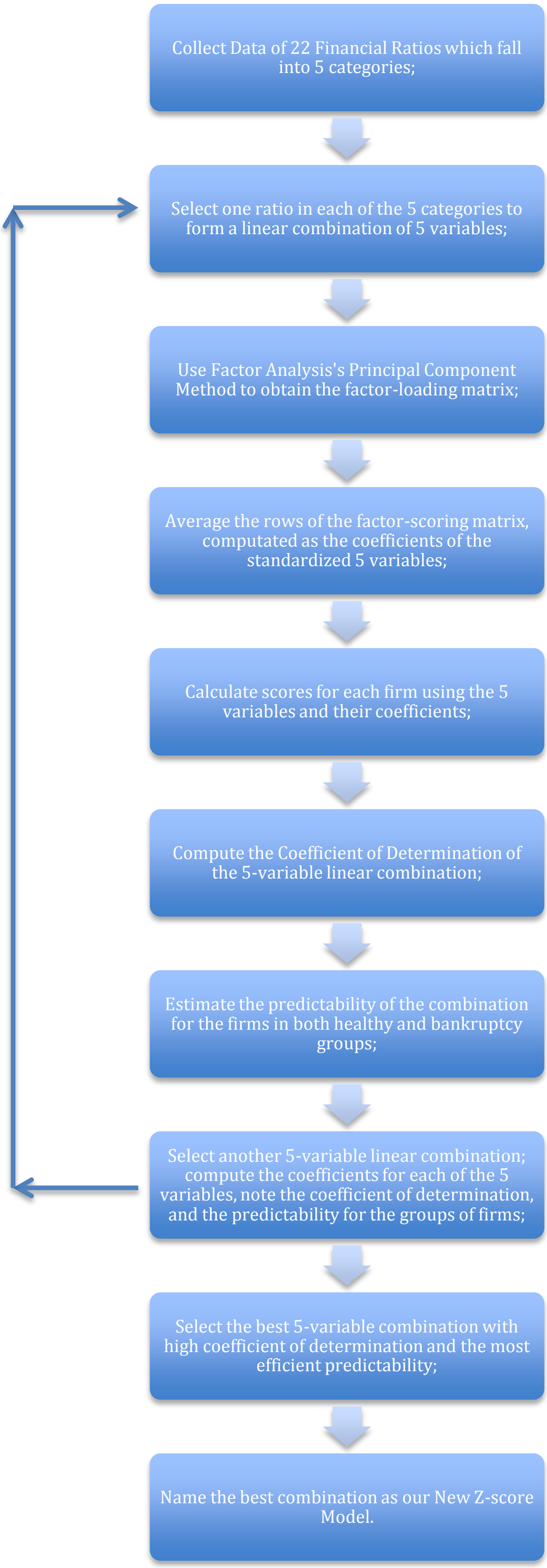
The sample data that are used in this study consists of two groups – the bankruptcy group and the healthy group. The data is collected through Bloomberg and UCLA-LoPucki Bankruptcy Research Database. The stock prices and financial ratios for healthy firms are manually collected from the Bloomberg; the financial ratios for the bankruptcy firms are manually collected from the links to the annual reports before bankruptcy declarations which provided by UCLA-LoPucki data base. The bankruptcy group has 33 companies that filed under Chapter 10K in U.S. government during the time period of 2003 to 2009. The healthy group has 33 companies that are still listed in NYSE (New York Stock Exchange) and NASDAQ stock markets. The financial ratios of bankruptcy companies are selected from their annual reports prior to bankruptcy announcements. The financial ratios in healthy group are selected from 2009 annual reports. The choice of using 2009 financial year considered that 2009 financial year is the most recent year with available data so it meets the up-to-date motivation of the study.

The reason of using financial ratios of combining data is explained in pervious section. Variable in ratios are effective tools for eliminating size effect among different companies. Therefore, when sampling the companies, sizes of them are not considered in this case. The financial ratios are selected under 5 categories -- liquidity, profitability, leverage, solvency, and activity. Those 22 ratios are the same as Altman's that were used in his study.

Chapter 4 Model Construction and Model Evaluation

Originally, we have 22 financial ratios, categorized under 5 categories.
After collecting data for the two groups, we have to select the best combination of ratios from the 5 categories that 1) can discriminate the two groups and 2) has high coefficient of determination. The criteria of selecting the best 5-variable linear combination don't have to be consistent, as long as the combination has the most efficient discrimination and acceptable high coefficient of determination. The combination has 5 variables (ratios), which should come from each of the financial categories.

The new Z-score model is obtained by using the data through factor analysis, principal component method. The software of doing this study is SPSS. In the factor model, 4 factors are selected as to construct a saturate factor model, whose factor-loading matrix is rotated in the fashion of maximum variation. The factor scores of those 4 factors are calculated under the method of regression. The flowchart of the analysis is as follows:


Collect Data of 22 Financial Ratios which fall into 5 categories;
Select one ratio in each of the 5 categories to form a linear combination of 5 variables;
Use Factor Analysis's Principal Component Method to obtain the factor-loading matrix;
Average the rows of the factor-scoring matrix, computated as the coefficients of the standardized 5 variables;
Calculate scores for each firm using the 5 variables and their coefficients;
Compute the Coefficient of Determination of the 5-variable linear combination;
Estimate the predictability of the combination for the firms in both healthy and bankruptcy groups;
Select another 5-variable linear combination; compute the coefficients for each of the 5 variables, note the coefficient of determination, and the predictability for the groups of firms;
Select the best 5-variable combination with high coefficient of determination and the most efficient predictability;
Name the best combination as our New Z-score Model.

The new Z-score of the factor model is calculated as the average of four factor scores, each factor score is regressed. Therefore, there is no need of showing the four-factor model but showing the average factor score model. The new Z-score model is as following:

$$Z = 0.035X_1 + 0.211X_2 + 0.142X_3 + 0.047X_4 + 0.202X_5 - 0.342$$

Where,

$X_1 = {}^{net\ working\ capital}/_{total\ assets}$

$X_2 = {}^{cumulative\ retained\ earnings}/_{total\ assets}$

$X_3 = EBIT/Total\ assets$

$X_4 = market\ value\ of\ equity/book\ value\ of\ total\ liabilities$

$X_5 = sales/total\ assets.$

Surprisingly, the ratios that were selected in the new Z-score model are identical with Altman's model. The new Z-score model has greater coefficient of determination than Altman's model. The following chart will demonstrate this characteristic.

Table 2 –Coefficient of Determination of the New Z-score Factor Model

Factor	Coefficient of Determination for Each Factor	Coefficient of Determination for Each Factor Model
1	43.286	43.286
2	21.694	64.981
3	18.461	83.442
4	10.430	93.872

Through the chart we can tell that the four-factor model can explain up to 93.872% of variation, i.e. equation 7. That is, the coefficient of determination of the factor model is 0.93872. The significant high coefficient of determination indicates that the factor model has great conviction.

After the computation, we get the parameters of the factor model. However, at this step, though we know the parameters of Altman's model, there is no way to know how well the Altman's model can explain the variation. Thus, the table below indicates the difference of the two models at this step:

Table 3 – Parameters of Altman's Model and the New Z-score Model

Parameters	Altman's Model	New Z-score Model
X_1	1.2	0.2825
X_2	1.4	0.30925
X_3	3.3	0.308
X_4	0.6	0.25
X_5	1.0	0.95
Cumulative % Variation	Unknown	93.872

With the new Z-score model, new Z-scores are calculated for every firm in each of the groups. The score of each firm is presented in **Appendix 3**. Theoretically, a positive new Z-score value implies a healthy firm and a negative new Z-score value implies a bankruptcy firm. Further predictability evaluation of both Altman's model and the new Z-score model will be presented in the following chapter.

Chapter 5 Predictability Evaluation

After having obtained the new model, its predictability needs to be evaluated. The evaluation is conducted the same was as the Altman's test table, i.e. Table 1, that is shown in early section of this study. Therefore, this study conducts two tests to find the predictabilities of Altman's model and the new Z-score model.

For Altman's model, misclassification will count mainly at Altman's Z-value rule of 1.81 -- "bankrupt" and 2.99 – "non-bankrupt" (Altman, 1968). There are 33 firms in the bankruptcy group and financial ratios are calculated through the last annual report before bankruptcy clearance; another 33 firms in the healthy group and financial ratios are calculated through the 2009 annual report. The predictability of Altman's model is the following:

Table 4 – Predictability of Altman's Model Using Updated Data

	Correct Number	% Correct	Total
Bankruptcy Group	27	81.8	33
Healthy Group	18	54.5	33
Total	45	68.2	66

For the new Z-score model, misclassification will count mainly at new Z-value rule of negative -- "bankrupt" and positive – "non-bankrupt". There are 33 firms in the bankruptcy group and financial ratios are calculated through the last annual report before bankruptcy clearance; another 33 firms in the healthy group and financial ratios are calculated through the 2009 annual report. The predictability of the new Z-score model is the following:

Table 5 -- Predictability of the New Z-score Model Using Updated Data

	Correct Number	% Correct	Total
Bankruptcy Group	27	81.8	33
Healthy Group	23	69.7	33
Total	50	75.8	66

So far, for Altman's Z-score model, the percentage of correction is obtained from the previous table. Thus, we can not only get the predictability accuracy of the model, but also get the coefficient of determination (R^2) of the model. Therefore, a full comparison between two models is built as the following table:

Table 6 – Predictability Comparison between Altman's Model and the New Z-score Model

	Altman's Model	New Z-score Model
% Correct on Bankruptcy Group	81.8	81.8
% Correct on Healthy Group	54.5	69.7
Coefficient of Determination	0.682	0.939

From the table above, the advantages of the new Z-score model are obvious. Though the new Z-score model's percentage of correct prediction for healthy group is same as Altman's model's; for the percentage of correct prediction on bankruptcy group, the new Z-score model has an advantage of 15.2% than that of Altman's model. For the coefficient of determination, the new Z-score model has 0.939 on the up-to-date data, which is 0.257 higher than that of Altman's model. Therefore, the new Z-score model has not only higher coefficient of determination, but also has more accuracy on the up-to-date data.

Chapter 6 Out-of-sample Comparisons

After obtaining the new Z-score model, a second data set was collected as out-of-sample to test if the new Z-score model's predictability continues to outperform with Altman's model.

The second data set contains 14 healthy companies listed in TSX (Toronto Stock Exchange) and 15 bankrupt companies. The data of the healthy group are financial ratios that are selected from 2011 annual reports; and the data of the bankrupt group are financial ratios that are selected from annual reports during the time period of 1997 to 2002. In the comparative tests, both Altman's Z-scores and the new Z-scores are calculated for these companies. For the healthy companies, any Altman's Z-scores lower than 2.99 are counted as prediction failure; and any new Z-scores lower than zero are counted as prediction failure. Also, for the bankruptcy companies, any Altman's Z-scores greater than 1.81 are counted as prediction failure; and any new Z-scores greater than zero are counted as prediction failure. The result of the test is shown in the following table.

Table 7 – Comparative Tests Using Out-of-sample Data

	Altman's Model	New Z-score Model
% Correct on Bankruptcy Group	80	80
% Correct on Healthy Group	71.4	100
Coefficient of Determination	0.757	0.939

From Table 7, the out-of-sample data test shows that the new Z-score model is still superior to Altman's model in term of prediction accuracy. Although the new Z-score model's percentage of correct prediction for bankruptcy group is same as Altman's model's, for the percentage of correct prediction on healthy group, the new Z-score model has an advantage of 29.6% over that of Altman's model (100% comes with the new Z-score model vs. 71.4% comes with Altman's model). For the coefficient of determination, the new Z-score model has 0.939 on the up-to-date data, which is 0.182 higher than that of Altman's model. Therefore, with the out-of-sample data, the new Z-score model has not only higher coefficient of determination, but also has more accuracy compared with Altman's model.

Chapter 7 Conclusion

Altman obtained his Z-score model in 1968. For more than 40 years, the model has been considered classical and accurate for predicting the bankruptcy possibilities of companies. This study's goal is to estimate another approach of factor analysis, with up-to-date data, to obtain the new Z-score model in order to improve Altman's model.

Compared with Altman's method in obtaining his model, factor analysis has its own advantages. The primary advantage of factor analysis is that the estimator is unbiased. It is crucially important. Usually the bias of traditional MDA (multivariate discriminate analysis) can't be avoided. The only way of reducing bias estimation of MDA is through reducing the sampling bias and searching bias. Factor analysis has unbiased estimator through the methodology itself. Another advantage of factor analysis is in the aspect of model evaluation. With completing of estimating the parameters, it will immediately get the percent of cumulative variance contribution, or the coefficient of determination. Researchers will immediately know how well the model in this aspect is. The traditional MDA method will only allow researchers to test the accuracy of the model and then get the coefficient of determination of the model. Factor analysis saves researchers much time and vigor on finding the coefficient of determination of the model.

Especially, the study finds that the coefficient of determination from factor analysis is much higher than Altman's method of MDA, which is 0.257 higher. Thus, the method of getting the new Z-score model, factor analysis, has advantages on unbiased estimator and immediately getting the higher coefficient of determination.

Apart from the advantages from the method itself, the new Z-score model has more advantages on predictabilities. Through the study, we find that the new Z-score model has its advantage on predicting the healthy firms. The overall percentage of correct prediction of the new Z-score is 7.6% higher than Altman's model.

Then, this study introduced another data set in order to test if the new Z-score model has its superiority. The results indicate that in both aspects, coefficient of determination and predictability, the new Z-score has more accuracy prediction on bankruptcy.

With a positive sign for non-bankruptcy and a negative sign for bankruptcy, some more works should be done in future research. This study only counts 66 firms from public stock markets. Thus, a larger database should be set and processed. In addition, factor analysis method is suggested to apply on study of private companies, unlisted companies and small-size companies.

Appendix 1 – Proof of Theorem 1 (He, 2012)

The proof of *Theorem* 1:

Based on the relations of R, λ_i, T, we have:

$$R = T\begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_P \end{pmatrix}T',$$

So, $R^{-1} = T\begin{pmatrix} \lambda_1^{-1} & 0 & \cdots & 0 \\ 0 & \lambda_2^{-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_P^{-1} \end{pmatrix}T'$

When R is irreversible, R^{-1} takes the generalized inverse matrix of R (Zhang and Fang, 1982),

In the form of matrix, $\Lambda^* = T_k\begin{pmatrix} \lambda_1^{-1} & 0 & \cdots & 0 \\ 0 & \lambda_2^{-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_P^{-1} \end{pmatrix}\Gamma_k,$

Where $T_k = (t_1, \dots, t_k)$.

Therefore, calculated by matrix multiplication, the Thompson factor score f^* of the regressor of Λ^*:

$$f^* = (\Lambda^*)' R^{-1} x$$

$$= \Gamma_k' \begin{pmatrix} \lambda_1^{-1/2} & 0 & \cdots & 0 \\ 0 & \lambda_2^{-1/2} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_P^{-1/2} \end{pmatrix} T_k' T_p \begin{pmatrix} \lambda_1^{-1} & 0 & \cdots & 0 \\ 0 & \lambda_2^{-1} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_P^{-1} \end{pmatrix} T_p' x$$

$$= \Gamma_k' \left(\lambda_1^{-1/2} t_1, \ldots, \lambda_k^{-1/2} t_k\right)' x = \Gamma_k' \left(\lambda_1^{-1/2} t_1' x_1, \ldots, \lambda_k^{-1/2} t_k' x_k\right)'$$

Since $\left(\lambda_1^{-1/2} t_1' x_1, \ldots, \lambda_k^{-1/2} t_k' x_k\right)'$ is the first k standardized principal components (Zhang and Fang, 1982), $\Gamma_k' \left(\lambda_1^{-1/2} t_1' x_1, \ldots, \lambda_k^{-1/2} t_k' x_k\right)' = f^*$ is the rotation of the first k standardized principal components. This completes the proof of ***Theorem* 1 i.** i.e. (6).

From (6), and the multiplication of matrices,

$$x = \Lambda^* f + u$$

Holds, and

$$\varepsilon(f^* | f) = \Gamma_k' \left(\lambda_1^{-1/2} t_1, \ldots, \lambda_k^{-1/2} t_k\right)' \varepsilon(\Lambda^* f + u | f) = \Gamma_k' \left(\lambda_1^{-1/2} t_1, \ldots, \lambda_k^{-1/2} t_k\right)' \Lambda^* f$$

$$= \Gamma_k' \left(\lambda_1^{-1/2} t_1, \dots, \lambda_k^{-1/2} t_k\right)' \left(\lambda_1^{-1/2} t_1, \dots, \lambda_k^{-1/2} t_k\right) \Gamma_k f = \Gamma_k' \Gamma_k f = f$$

That is, the Thompson factor score is unbiased, and its average prediction error is smaller (Fang, 1989). This completes the proof of ***Theorem* 1 ii.**

Appendix 2 – Proof of Theorem 2 (He, 2012)

For proving **Theorem 2**, we need to introduce **Lemma 1**;

Lemma 1 (Weyl's Lemma**):** Let both R and B be p-order symmetric matrices, where the eigenvalues for R are $\lambda_1, \dots, \lambda_p$,

and

$$\lambda_1 \geq \cdots \geq \lambda_p$$

For B are $v_1, \dots, v_p$,

$$v_1 \geq \cdots \geq v_p.$$

If R-B is non-negative definite, then

$$\lambda_j \geq v_j \ (j = 1, \dots, p).$$

Theorem 2 proof： For any matrix of factor loadings Λ, $\mathrm{R} = \Lambda\Lambda' + \Psi$, So the difference,

$$\mathrm{R} - \Lambda\Lambda' = \Psi \geq 0.$$

That is, Ψ is a non-negative definite matrix.

Let $v_1, \dots, v_p, v_j = 0\ (j > k)$ be eigenvalues of $\Lambda\Lambda'$, where $v_1 \geq \cdots \geq v_p \geq 0$.

According to the assumption of R and its eigenvalues λ_j, and from *Lemma* 1,

$\lambda_j \geq v_j\ (j = 1, \dots, p)$ holds. Hence

$$tr(\Lambda\Lambda') = \sum v_j \leq \sum_{j=1}^{k} \lambda_j.$$

This completes the proof of ***Theorem* 2 (i),** i.e. (6).

From the spectral decomposition of R: $R = \sum \lambda_j t_j t_j'$, Calculated by multiplication of the matrix: $\Lambda^*(\Lambda^*)' = \sum \lambda_j t_j t_j'$. We have

$R - \Lambda^*(\Lambda^*)' = \sum_{j=k+1}^{r} \lambda_j t_j t_j'$, and $\lambda_j > 0\ (j = k+1, \dots, r)$, Also

$R - \Lambda^*(\Lambda^*)'$ is non-negative definite,

And

$$\sum_{j=1}^{k} q_j = \sum_{i=1}^{p} h_i^2 = \sum_{i=1}^{p} \sum_{j=1}^{k} \lambda_{ij}^2 = tr(\Lambda'\Lambda) = tr(\Lambda\Lambda').$$

This completes the proof of ***Theorem* 2 (ii)**, i.e. (7).

Appendix 3 – Altman's Scores and Factor Scores of the Data (first 33 companies belong to bankruptcy group; second 33 companies belong to healthy group)

Company	Date	Altman's Z	New Z
aaiPharma	04-12-31	-2.864710975	-0.820775
Abitibibowater Inc.	07-12-31	0.238907221	-0.298735
Accuride Corporation	08-12-31	-0.234490702	-0.33594
Amcast Industrial Corporation	03-8-31	1.595279958	-0.041435
Anchor Glass Container Corporation	04-12-31	1.08282922	-0.13578
Asyst Technologies	08-3-31	0.103414633	-0.376185
Calpine Corporation	04-12-31	0.477004841	-0.254535
Caraustar Industries, Inc.	08-12-31	0.311031365	-0.21598
Champion Enterprises, Inc.	09-1-3	1.041850871	-0.133835
Chesapeake Corporation	07-12-30	1.681246587	-0.08738
Circuit City Stores, Inc.	08-2-29	3.638979854	0.374615
Collins & Aikman Corporation	03-12-31	0.966047965	-0.166135
Dana Corporation	04-12-31	1.748336512	-0.027715
Acterna	03-3-31	-8.294386933	-1.6955725

Corporation			
Delta Air Lines	04-12-31	-0.198251799	-0.35459
Falcon Products	03-11-1	1.440458923	-0.0641125
General Motors Corporation	08-12-31	-0.885750572	-0.474335
Granite Broadcasting Corporation	05-12-31	0.539823903	-0.2447775
GSI Group Inc.	07-12-31	12.44504967	0.49578
Hancock Fabrics	06-1-28	3.113376974	0.25597
Hines Horticluture, Inc.	06-12-31	0.391225329	-0.2789075
Inphonic	06-12-31	0.967552775	-0.31907
Lear Corporation	08-12-31	1.393011345	-0.0441875
AMR Corporation	10-12-31	0.754494776	-0.2138825
Lyondell Chemical Company	07-12-31	1.466897093	-0.0960225
McLeodUSA Incorporated	04-12-31	-2.593675906	-0.7879475
Northwest Airlines Corporation	04-12-31	42.95090498	1.75601
Applied Extrusion Technologies, Inc.	03-9-30	0.972188541	-0.1624575
Silicon Graphics	05-6-24	-3.998750008	-1.2026925
Tweeter Home Entertainment Group	06-9-30	1.702220544	-0.00619
Ultimate Electronics	04-1-31	2.839541561	0.1634025
Visteon Corporation	08-12-31	0.988469626	-0.1203075
Winn-Dixie Stores	04-6-30	4.929533647	0.5985475
Amcon Distributing	09-9-30	11.93532197	1.9647875
AAR Corp.	09-5-31	2.700031934	0.0941825
Abercrombie & Fitch Co.	09-1-31	4.101159493	0.30714
B & G Foods, Inc.	09-1-3	1.202269581	-0.149815
Best Buy Co.	09-2-28	4.265865813	0.3978875
The Boeing Company	09-12-31	2.179823027	0.0303775
Cabot Corp	09-9-30	2.161396681	0.0021
Cambrex Corporation	09-12-31	1.72281814	-0.079835
Danaher	09-12-31	3.415065493	0.050785

Corporation			
Diebold Incorporated	09-12-31	2.959142031	0.11954
Eagle Materials, Inc.	09-3-31	2.42734471	-0.01688
Ennis, Inc.	09-2-28	3.11515299	0.1488975
Feihe International Inc	09-12-31	2.035675987	-0.1071775
Ferro Corporation	09-12-31	2.018587529	-0.00271
Gardner Denver, Inc.	09-12-31	2.861685314	0.0102875
Glatfelter	09-12-31	3.113811757	0.180285
H.J. Heinz Company	09-4-29	3.34959136	0.19203
Harley-Davidson, Inc.	09-12-31	2.334024714	0.046765
Harman International	09-6-30	2.017293337	0.0172875
James Hardie Industries SE	09-3-31	2.552081032	-0.1267425
Kaydon Corporation	09-12-31	9.297301829	0.482845
Kraft Foods Inc	09-12-31	1.752615515	-0.0931175
Lennox International	09-12-31	4.172020209	0.2668375
Lockheed Martin Corporation	09-12-31	2.758896786	0.1073525
Lydall, Inc.	09-12-31	6.984412516	0.7195225
Magna International Inc	09-12-31	2.453047329	0.049935
Manitowox Company Inc.	09-12-31	0.772455316	-0.197085
Masco Corporation	09-12-31	1.802749285	-0.0526975
McDonald's	09-12-31	5.455448202	0.340565
Medtronic, Inc.	09-4-24	3.875143982	0.1242175
Nike, Inc.	09-5-31	6.741443761	0.4116625
Omega Protein Corporation	09-12-31	2.141990412	-0.039285
Pall Corporation	09-7-31	3.325965918	0.1152

Appendix 4 – Out-of-sample Data Set (first 15 companies belong to bankruptcy group; second 14 companies belong to healthy group)

Company	Altman's Z score	New Z-score
1	0.268897538	-0.119975985
2	2.132201461	-0.057574296
3	0.495111843	-0.09959739
4	1.51850449	-0.194484323
5	-0.072820019	-0.240006514
6	2.239233387	0.01728559
7	1.898126806	0.199140299
8	-1.545008912	-0.1150304
9	-1.345764319	-0.038363668
10	0.409539919	-0.066329297
11	0.492941359	-0.110935315
12	0.040539293	-0.015500051

13	1.463590746	-0.139638436
14	0.264772548	-0.158275364
15	0.097960937	0.014614311
16	-0.547408161	0.115717118
17	0.053009097	0.123121395
18	1.202909596	0.500218482
19	2.422891499	0.915936319
20	-1.847538771	0.129687593
21	0.050519653	0.196136979
22	0.040448758	0.319110979
23	4.719809871	1.687358276
24	2.486587089	0.800527521
25	3.772358225	1.272617755
26	3.675777755	1.44115538
27	1.430232	0.62743955
28	7.822424194	3.706475928
29	2.147060642	0.829638601

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