Hopf Algebras

Definitions and Examples

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Abstract

We define a Hopf algebra and give a variety of examples of varying complexity. To facilitate the definition, we first define the commutative diagram, the tensor product, and an algebra/coalgebra/bialgebra. We briefly discuss the duality between algebras and coalgebras. Prior to introducing the non-commutative Hopf algebras of Sweedler and Taft, we define the q-binomial coefficient and prove a related lemma from q-series which allows an explicit formula for the coproduct of a Taft algebra.

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1. Introduction

A Hopf algebra is a special algebraic construction which arises in associative algebra, algebraic topology, combinatorics, and quantum physics. Hopf algebras are named after Heinz Hopf, a mathematician who worked in algebraic topology in the 1940s. In 1941, Hopf published a paper discussing a certain type of manifold and the special structure of its cohomology ring. Topologists began to call rings with this structure "Hopf algebras" in the 1950s. John Milnor and John Moore coauthored a paper in 1965, in which their definition of Hopf algebra coincides with the modern definition of a graded bialgebra [1]. The modern definition of Hopf algebra (a bialgebra with antipode) was given by Moss Sweedler [10] in his book from 1969.

In the 1960s, algebraists began to study Hopf algebras separate from their topological application. Sweedler and Earl Taft [11] introduced the first examples of noncommutative noncocommutative Hopf algebras, using ideas from q-series, around 1970. Irving Kaplansky gave a set of ten conjectures on Hopf algebras in 1973, some of which have been proven or disproven, and some of which remain open [9]. A large part of the current algebraic research on Hopf algebras is classification; for example, Nicolas Andruskiewitsch and Hans-Jurgen Schneider [2] proposed a method of classifying pointed Hopf algebras via filtrations and gradings.

Hopf algebras became important in quantum physics due to an influential paper of Vladimir Drinfeld [4] in 1988. Drinfeld defined a quantum group to be the spectrum of a Hopf algebra and expressed his interest in non-commutative Hopf algebras. This led to generalizations of Hopf algebras in the setting of braided monoidal categories [6].

With the recent developments in quantum computing, researchers have noticed that category theory and Hopf algebras can characterize the structure of quantum circuits. For example, Naoki Sasakura [8] wrote a paper in 2010 showing that any quantum circuit can be described in terms of the operations of a Hopf algebra and unitary transformations on a single qubit. We will describe this construction in more detail in the section defining Hopf algebras.

Before proceeding to the linear algebraic foundations of Hopf algebra theory, let us establish some conventions. We will usually omit reference to the scalar field of vector spaces, assuming that all vector spaces considered are over the same field $\Bbbk$. (Occasionally we will refer to $\Bbbk$, especially in definitions and when we require that $\Bbbk$ have certain properties). Rather than give an explicit formula when defining linear maps, we will usually define their action on a basis of their domain (such as for our numerous examples of algebras, coalgebras, bialgebras, and Hopf algebras) or, in the case of a linear map on tensors, on the spanning set of simple tensors. Similarly, we will often define algebra homomorphisms $A \to B$ on a generating set of A for the same reason. The reader may assume or verify that there is no contradiction derived from these definitions. When defining a "canonical" monomorphism or isomorphism (in particular the examples in the section on tensor products), we will name it only when convenient.

2. Commutative Diagram

Category theory is an axiom system alternative to set theory. In the following definition we will use the word "collection" and the symbol $\in$ with the understanding that a collection can be either a set or a proper class.

Definition (Category)**.** *A category $\mathcal{C}$ consists of a collection of objects $\mathcal{C}_0$ and arrows $\mathcal{C}_1$ satisfying the following axioms.*

- *Each arrow f is associated with a pair of objects $\mathrm{dom}(f)$ and $\mathrm{cod}(f)$. If we label $\mathrm{dom}(f) = x$ and $\mathrm{cod}(f) = y$, we typically write $f : x \to y$ and $f \in \mathrm{Hom}(x, y)$.*
- *Each pair of arrows of the form $f : x \to y$ and $g : y \to z$ corresponds to a composite arrow $g \circ f : x \to z$.*
- *Each object x corresponds to an identity arrow $\mathrm{id}_x : x \to x$ such that for each arrow of the form $f : x \to y$, we have $f \circ \mathrm{id}_x = f$; and for each arrow of the form $f : y \to x$, we have $\mathrm{id}_x \circ f = f$.*
- *For each triple of arrows of the form $f : w \to x$, $g : x \to y$, and $h : y \to z$, we have $h \circ (g \circ f) = (h \circ g) \circ f$.*

The most obvious example of a category is **Set**, whose objects are sets and whose arrows are functions. Other important categories are $\mathbf{Vect}_{\Bbbk}$ ($\Bbbk$-vector spaces and linear maps), $\mathbf{Algebra}_{\Bbbk}$ ($\Bbbk$-algebras and algebra homomorphisms), and ${}_A\mathbf{Module}$ (left A-modules and left A-module homomorphisms).

We will not go far into category theory. Our main purpose in this section is to introduce commutative diagrams. The formal definition of a commutative diagram requires too much category-theoretical background, so we will give a definition which emphasizes the graphical nature of a diagram.

Definition (Commutative Diagram)**.** *A commutative diagram is a directed graph which represents a functional equation. The vertices of the graph are objects and the edges are arrows (where the codomain is the vertex pointed to by the edge). If two different paths through the graph have the same start and end point, the compositions of the arrows corresponding to the edges are equal.*

Example 2.1. The following diagram represents Euler's identity.

$$\begin{tikzcd} \mathbb{R} \arrow[r, "ix"] \arrow[dr, "\cos(x)+i\sin(x)"'] & \mathbb{C} \arrow[d, "e^x"] \\ & \mathbb{C}\setminus\{0\} \end{tikzcd}$$

It is very common to describe functional equations in algebra with commutative diagrams. We will do so in definitions but also write the functional equations explicitly for ease of reading.

3. Tensor Product

Definition (Free Vector Space)**.** *Let X be a set. The free vector space on X, denoted $\Bbbk X$, is the unique vector space (up to isomorphism) with an injective map $i : X \to \Bbbk X$ such that the image of i is a basis of $\Bbbk X$.*

Definition (Tensor Product)**.** *Let U, V be vector spaces and let R be the subspace of $\Bbbk(U \times V)$ spanned by elements of the form*

$$\begin{aligned} i(a+b, v) - i(a, v) - i(b, v) \\ i(u, a+b) - i(u, a) - i(u, b) \\ i(\lambda u, v) - \lambda i(u, v) \\ i(u, \lambda v) - \lambda i(u, v) \end{aligned}$$

The tensor product of U and V, denoted by $U \otimes V$, is the quotient space $\Bbbk(U \times V)/R$. Elements in $U \otimes V$ of the form $i(u, v) + R$ are denoted $u \otimes v$.

Lemma 3.1. *If A and B are bases for U and V respectively, then $\{a \otimes b : a \in A, b \in B\}$ is a basis for $U \otimes V$.*

Corollary 3.2. *Every vector in $U \otimes V$ is a linear combination of elements of the form $u \otimes v$.*

Corollary 3.3. $\dim(U \otimes V) = \dim(U)\dim(V)$

Remark 3.4. Elements of $U \otimes V$ of the form $u \otimes v$ are called simple tensors, but not every element of $U \otimes V$ is a simple tensor. Every simple tensor $u \otimes v$ can be written $(\lambda u) \otimes (\lambda^{-1} v)$ for all non-zero $\lambda \in \Bbbk$, so the projection of $i(U \times V)$ is neither injective nor surjective. It is also not linear, but it is bilinear.

Definition (Bilinear Map)**.** *Let U, V, W be vector spaces. A bilinear map $f : U \times V \to W$ is a map satisfying*

$$f(u_1 + u_2, v) = f(u_1, v) + f(u_2, v)$$

$$f(u, v_1, v_2) = f(u, v_1) + f(u, v_2)$$

$$f(\lambda u, v) = \lambda f(u, v) = f(u, \lambda v)$$

for all $u_1, u_2, u \in V, v_1, v_2, v \in V, \lambda \in \Bbbk$.

As we noted in the introduction, we will usually define linear maps $U \otimes V \to W$ on the spanning set of simple tensors, and the reader may assume that such maps are well-defined.

Definition (Flip Map)**.** *The flip map $\tau : U \otimes V \to V \otimes U$ is the isomorphism satisfying $\tau(u \otimes v) = v \otimes u$.*

Remark 3.5. Sweedler [10, p. 49] calls the flip map the twist map, but modern usage of "twist" in algebra usually refers to parametrized deformations of Hopf algebras.

There are some additional canonical isomorphisms and monomorphisms which we will use frequently.

Example 3.6. Since $\Bbbk$ is a 1-dimensional vector space,

$$U \cong U \otimes \Bbbk \cong \Bbbk \otimes U$$

The canonical isomorphisms take u to $u \otimes \lambda$ and $\lambda \otimes u$ respectively. In particular,

$$\Bbbk \otimes \Bbbk \cong \Bbbk$$

The canonical isomorphism takes $\lambda \otimes \mu$ to $\lambda\mu$.

Example 3.7. There is a canonical monomorphism $\phi : \mathrm{Hom}(U, U') \otimes \mathrm{Hom}(V, V') \to \mathrm{Hom}(U \otimes V, U' \otimes V')$ satisfying

$$\phi(f \otimes g)(u \otimes v) = f(u) \otimes g(v)$$

In practice we usually denote $\phi(f \otimes g)$ by $f \otimes g$ for convenience.

Example 3.8. There is a canonical monomorphism $\phi : U^* \otimes U^* \to (U \otimes U)^*$ satisfying

$$\phi(f \otimes g)(u \otimes v) = f(u)g(v)$$

If U is finite-dimensional, then ϕ is an isomorphism [7, p. 378].

Example 3.9. There is a canonical isomorphism $(U \otimes V) \otimes W \to U \otimes (V \otimes W)$ taking $(u \otimes v) \otimes w$ to $u \otimes (v \otimes w)$. This can be extended to any number of vector spaces. In practice we usually identify both $(U \otimes V) \otimes W$ and $U \otimes (V \otimes W)$ by $U \otimes V \otimes W$ and so on.

Definition (n-fold Tensor Product)**.** *The n-fold tensor product of U is defined recursively by*

$$U^{\otimes 0} = \mathbb{k}$$

$$U^{\otimes n+1} = U \otimes U^{\otimes n}$$

Remark 3.10. Given $f : U \to V$, we define $f^{\otimes n} : U^{\otimes n} \to V^{\otimes n}$ in the obvious way.

4. Unital Associative Algebra

Definition (Unital Associative Algebra)**.** *A unital associative* $\mathbb{k}$*-algebra* (A, ∇, η) *is a* $\mathbb{k}$*-vector space* A *with a linear product map* $\nabla : A \otimes A \to A$ *and a linear unit map* $\eta : \mathbb{k} \to A$ *such that the following diagrams commute.*

$$\begin{tikzcd}
& A \otimes A \otimes A \arrow[dl, "\nabla\otimes\mathrm{id}"'] \arrow[dr, "\mathrm{id}\otimes\nabla"] & \\
A \otimes A \arrow[dr, "\nabla"] & & A \otimes A \arrow[dl, "\nabla"] \\
& A &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{k} \otimes A \arrow[r, "\cong"] \arrow[dr, "\eta\otimes\mathrm{id}"] & A & A \otimes \mathbb{k} \arrow[l, "\cong"'] \arrow[dl, "\mathrm{id}\otimes\eta"] \\
& A \otimes A \arrow[u, "\nabla"] &
\end{tikzcd}$$

Remark 4.1. The relations represented by the diagrams are

$$\nabla(\nabla(x \otimes y) \otimes z) = \nabla(x \otimes \nabla(y \otimes z))$$

$$\nabla(\eta(\lambda) \otimes x) = \nabla(x \otimes \eta(\lambda)) = \lambda x$$

Given a vector space A, a product map and unit map induce the structure of a unital ring on A and vice versa. We define $xy = \nabla(x \otimes y)$ and $1_A = \eta(1_{\mathbb{k}})$.

We will shorten (A, ∇, η) as A, "unital associative algebra" to "algebra", $\nabla(x \otimes y)$ to xy, and $\eta(\lambda)$ to λ when convenient. Let us proceed to some definitions.

Definition (Algebra Homomorphism)**.** *Given two algebras A and B, an algebra homomorphism $f : A \to B$ is a linear map satisfying*

$$f(xy) = f(x)f(y)$$
$$f(1_A) = 1_B$$

for all $x, y \in A$.

Definition (Algebra Antimorphism)**.** *Given two algebras A and B, an algebra antimorphism $f : A \to B$ is a linear map satisfying*

$$f(xy) = f(y)f(x)$$
$$f(1_A) = 1_B$$

for all $x, y \in A$.

Remark 4.2. We will use the definition of algebra homomorphism in a lemma on bialgebras and the definition of algebra antimorphism in the definition of the Hopf algebra antipode.

Example 4.3 (Field)*.* A field $\Bbbk$ is an algebra over any of its subfields.

Example 4.4 (Free Algebra)*.* The free algebra in $\{x_1, \cdots, x_n\}$ is the non-commutative polynomial ring $\Bbbk\langle x_1, \cdots, x_n\rangle$.

Example 4.5 (Matrix Algebra)*.* The matrix ring $M_{n\times n}(\Bbbk)$ is an algebra. The unit map is given by $\eta(1) = I$, where I is the identity matrix.

Example 4.6 (Monoid Algebra). Let M be a monoid. The monoid algebra on $\Bbbk M$ has product and unit given by $i(m)i(n) = i(mn)$ for all $m, n \in M$ (where i is the canonical embedding of M into $\Bbbk M$) and $\eta(1) = i(1_M)$.

Definition (Left Module)**.** *Let A be a $\Bbbk$-algebra. A left A-module (M, m) is a $\Bbbk$-vector space equipped with an action $m : A \otimes M \to M$ such that the following diagrams commute.*

$$\begin{tikzcd}
\Bbbk \otimes M \arrow[r, "\eta\otimes\mathrm{id}"] \arrow[dr, "\cong"] & A \otimes M \arrow[d, "m"] \\
& M
\end{tikzcd}$$

$$\begin{tikzcd}
& A \otimes A \otimes M \arrow[dl, "\nabla\otimes\mathrm{id}"'] \arrow[dr, "\mathrm{id}\otimes m"] & \\
A \otimes M \arrow[dr, "m"] & & A \otimes M \arrow[dl, "m"] \\
& M &
\end{tikzcd}$$

Remark 4.7. The relations represented by the diagrams are

$$m(\eta(\lambda) \otimes x) = \lambda x$$

$$m(\nabla(a \otimes b) \otimes x) = m(a \otimes m(b \otimes x))$$

A right A-module M is defined in a parallel manner, with $m : M \otimes A \to M$ etc. We have chosen to work with left A-modules because the notation is closer to the familiar scalar multiplication from linear algebra.

We will shorten (M, m) to M and $m(a \otimes x)$ to ax when convenient.

5. Coalgebra

Definition (Counital Coassociative Coalgebra)**.** *A counital coassociative* $\mathbb{k}$*-coalgebra* (C, Δ, ϵ) *is a* $\mathbb{k}$*-vector space* C *with a linear coproduct map* $\Delta : C \to C \otimes C$ *and a linear counit map* $\epsilon : C \to \mathbb{k}$ *such that the following diagrams commute.*

$$\begin{tikzcd}
& C \arrow[dl, "\Delta"'] \arrow[dr, "\Delta"] & \\
C \otimes C \arrow[dr, "\Delta\otimes\mathrm{id}"] & & C \otimes C \arrow[dl, "\mathrm{id}\otimes\Delta"] \\
& C \otimes C \otimes C &
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{k} \otimes C & C \arrow[l, "\cong"'] \arrow[r, "\cong"] \arrow[d, "\Delta"] & C \otimes \mathbb{k} \\
& C \otimes C \arrow[ul, "\epsilon\otimes\mathrm{id}"] \arrow[ur, "\mathrm{id}\otimes\epsilon"'] &
\end{tikzcd}$$

Remark 5.1. The relations represented by the diagrams are

$$(\Delta \otimes \mathrm{id})(\Delta(x)) = (\mathrm{id} \otimes \Delta)(\Delta(x))$$

$$(\epsilon \otimes \mathrm{id})(\Delta(x)) = 1 \otimes x$$

$$(\mathrm{id} \otimes \epsilon)(\Delta(x)) = x \otimes 1$$

These diagrams are categorically dual to the diagrams for an algebra. The property encoded by the first diagram is called coassociativity.

We will shorten (C, Δ, ϵ) to C and "counital coassociative coalgebra" to "coalgebra" when convenient. Let us now introduce the n-fold coproduct and the Sweedler notation.

Definition (n-fold Coproduct)**.** *The n-fold coproduct $\Delta_n : C \to C^{\otimes n+1}$ is defined for each n recursively by*

$$\Delta_1 = \Delta$$

$$\Delta_{n+1} = (\Delta \otimes \mathrm{id}^{\otimes n}) \circ \Delta_n$$

Lemma 5.2. *Let $n \geq 2$ and $0 \leq j \leq n$. Then*

$$\Delta_{n+1} = (\mathrm{id}^{\otimes j} \otimes \Delta \otimes \mathrm{id}^{\otimes n-j}) \circ \Delta_n$$

Sweedler [10, p. 10] suggests the notation

$$\Delta(x) = \sum_{(x)} x_{(1)} \otimes x_{(2)}$$

$$(f \otimes g)(\Delta(x)) = \sum_{(x)} f(x_{(1)}) \otimes g(x_{(1)})$$

and in general

$$\Delta_n(x) = \sum_{(x)} x_{(1)} \otimes \cdots \otimes x_{(n+1)}$$

$$(f_1 \otimes \cdots \otimes f_{n+1})(\Delta_n(x)) = \sum_{(x)} f_1(x_{(1)}) \otimes \cdots \otimes f_{n+1}(x_{(n+1)})$$

This notation is widely used (sometimes even without a summation symbol) and makes it much more convenient to describe functional equations containing coproducts.

Definition (Subcoalgebra)**.** *A subcoalgebra V of C is a subspace of C such that $\Delta(V) \subseteq V \otimes V$.*

Definition (Left, Right Coideal)**.** *A left (resp. right) coideal I of C is a subspace of C such that $\Delta(I) \subseteq C \otimes I$ (resp. $\Delta(I) \subseteq I \otimes C$).*

Definition (Two-sided Coideal)**.** *A two-sided coideal I of C is a subspace of C such that $\Delta(I) \subseteq C \otimes I + I \otimes C$ and $\epsilon(I) = 0$.*

Remark 5.3. There are notions of subalgebras and ideals of algebras, but they are simply the equivalent definitions in ring theory. We did not include them in the previous section because it would make the text too dense.

Definition (Coalgebra Homomorphism)**.** *Given two coalgebras C and D, a coalgebra homomorphism $f : C \to D$ is a linear map satisfying*

$$\Delta_D(f(x)) = \sum_{(x)} f(x_{(1)}) \otimes f(x_{(2)})$$

$$\epsilon_D(f(x)) = \epsilon_C(x)$$

for all $x \in C$.

Definition (Coalgebra Antimorphism)**.** *Given two coalgebras C and D, a coalgebra antimorphism $f : C \to D$ is a linear map satisfying*

$$\Delta_D(f(x)) = \sum_{(x)} f(x_{(2)}) \otimes f(x_{(1)})$$
$$\epsilon_D(f(x)) = \epsilon_C(x)$$

for all $x \in C$.

Remark 5.4. Just as before, we use these notions of coalgebra homomorphism and antimorphism in the sections on bialgebras and Hopf algebras respectively.

Definition (Cocommutative Coalgebra)**.** *C is said to be cocommutative if Δ satisfies*

$$\sum_{(x)} x_{(1)} \otimes x_{(2)} = \sum_{(x)} x_{(2)} \otimes x_{(1)}$$

for all $x \in C$.

Example 5.5 (Field)*.* A field $\Bbbk$ has coalgebra structure, with coproduct and counit defined $\Delta(\lambda) = 1 \otimes \lambda$ and $\epsilon(\lambda) = \lambda$.

Example 5.6 (Polynomial Coalgebra). The set of polynomials $\Bbbk[x]$ has coalgebra structure, where the coproduct and counit are given by

$$\Delta(x^n) = \sum_{j=0}^{n} \begin{pmatrix} n \\ j \end{pmatrix} x^j \otimes x^{n-j}$$

$$\epsilon(x^n) = \begin{cases} 1 & n = 0 \\ 0 & n \geq 1 \end{cases}$$

Example 5.7 (Matrix Coalgebra). The set of $n \times n$ matrices $M_{n\times n}(\Bbbk)$ has coalgebra structure. Let $\{e_{jk} : 1 \leq j, k \leq n\}$ be the canonical basis for $M_{n\times n}(\Bbbk)$. The coproduct and counit are given by

$$\Delta(e_{jk}) = \sum_{\ell=1}^{n} e_{j\ell} \otimes e_{\ell k}$$

$$\epsilon(e_{jk}) = \begin{cases} 1 & j = k \\ 0 & j \neq k \end{cases}$$

Example 5.8 (Grouplike Coalgebra). Let X be a set. The grouplike coalgebra (sometimes free pointed coalgebra) on $\Bbbk X$ has coproduct and counit given by $\Delta(x) = x \otimes x$ and $\epsilon(x) = 1$ for all $x \in X$.

Definition (Grouplike Element). *Let C be any coalgebra. A grouplike element $x \in C$ is an element satisfying $\Delta(x) = x \otimes x$ and $\epsilon(x) = 1$.*

Example 5.9 (Trigonometric Coalgebra)*.* Let $T = \{c, s\}$. The trigonometric coalgebra on $\Bbbk T$ has coproduct and counit given by

$$\Delta(c) = c \otimes c - s \otimes s$$

$$\Delta(s) = c \otimes s + s \otimes c$$

$$\epsilon(c) = 1$$

$$\epsilon(s) = 0$$

Definition (Right Comodule)**.** *Let C be a $\Bbbk$-coalgebra. A right C-comodule (M, c) is a $\Bbbk$-vector space equipped with an linear structure map $c : M \to M \otimes C$ such that the following diagrams commute.*

$$\begin{tikzcd} M \arrow[r, "c"] \arrow[dr, "\cong"] & M \otimes C \arrow[d, "\mathrm{id} \otimes \epsilon"] \\ & M \otimes \Bbbk \end{tikzcd}$$

$$\begin{tikzcd} & M \arrow[dl, "c"'] \arrow[dr, "c"] & \\ M \otimes C \arrow[dr, "c \otimes \mathrm{id}"'] & & M \otimes C \arrow[dl, "\mathrm{id} \otimes \Delta"] \\ & M \otimes C \otimes C & \end{tikzcd}$$

Remark 5.10*.* The relations represented by the diagrams are

$$(\mathrm{id} \otimes \epsilon)(c(x)) = x \otimes 1$$

$$(c \otimes \mathrm{id})(c(x)) = (\mathrm{id} \otimes \Delta)(c(x))$$

Remark 5.11. A left C-comodule is defined in a parallel manner, with $c : M \to C \otimes M$ etc. We have chosen to work with right C-comodules because of the duality with left C^*-modules (described later) and because the Sweedler notation for right C-modules is more convenient.

We will shorten (M, c) to M when convenient.

Definition (n-fold Comodule Structure Map)**.** *The n-fold comodule structure map $c_n : M \to M \otimes C^{\otimes n}$ is defined for each n recursively by*

$$c_1 = c$$

$$c_{n+1} = (c \otimes \mathrm{id}^{\otimes n}) \circ c_n$$

Lemma 5.12. *Let $n \geq 2$ and $1 \leq j \leq n$. Then*

$$c_{n+1} = (\mathrm{id}^{\otimes j} \otimes \Delta \otimes \mathrm{id}^{\otimes n-j}) \circ c_n$$

Sweedler [10, p. 32] extends his notation for coalgebras to comodules, suggesting the notation

$$c(x) = \sum_{(x)} x_{(0)} \otimes x_{(1)}$$

$$(f \otimes g)(c(x)) = \sum_{(x)} f(x_{(0)}) \otimes g(x_{(1)})$$

and in general

$$c_n(x) = \sum_{(x)} x_{(0)} \otimes \cdots \otimes x_{(n)}$$

$$(f \otimes g_1 \otimes \cdots \otimes g_n) \circ c_n(x) = \sum_{(x)} f(x_{(0)}) \otimes g_1(x_{(1)}) \otimes \cdots \otimes g_n(x_{(n)})$$

Note that the first symbol in the tensor product is indexed by 0 rather than 1. This is because it is an element of M while the others are elements of C.

6. Duality

Definition (Convolution Algebra)**.** *Let A be an algebra and C a coalgebra. The convolution algebra on* $\mathrm{Hom}(C, A)$ *satisfies* $\eta(1)(x) = \epsilon(x)$ *and*

$$\nabla(f \otimes g)(x) = \sum_{(x)} f(x_{(1)})g(x_{(2)})$$

Remark 6.1. The product of two functions f, g in the convolution algebra is called the convolution of f and g and typically written $f * g$.

Definition (Dual Algebra)**.** *Let C be a coalgebra. The dual algebra of C is the convolution algebra on $C^* = \mathrm{Hom}(C, A)$.*

Lemma 6.2. *Let A be a finite-dimensional algebra. Then for each $f \in A^*$, there exists a unique linear map in $A^* \otimes A^*$*

$$\sum_{j=1}^{n} g_j \otimes h_j$$

such that

$$f(xy) = \sum_{j=1}^{n} g_j(x)h_j(y)$$

for all $x, y \in A$.

Proof. Recall from Example 3.8 that the canonical monomorphism $\phi : A^* \otimes A^* \to (A \otimes A)^*$ is an isomorphism because A is finite-dimensional.

Then write

$$\phi^{-1}(f \circ \nabla) = \sum_{j=1}^{n} g_j \otimes h_j$$

It follows from definition of ϕ that

$$f(xy) = \sum_{j=1}^{n} g_j(x) h_j(y)$$

for all $x, y \in A$ as desired. $\square$

Definition (Dual Coalgebra)**.** *Let A be a finite-dimensional algebra. The dual coalgebra of A is the coalgebra on A^* satisfying $\Delta(f) = \phi^{-1}(f \circ \nabla)$ and $\epsilon(f) = f(1)$.*

7. Bialgebra

Definition (Bialgebra)**.** *A $\mathbb{k}$-bialgebra $(B, \nabla, \Delta, \eta, \epsilon)$ is a $\mathbb{k}$-vector space with algebra and coalgebra structure such that the following diagrams commute.*

$$\begin{tikzcd}
B \otimes B \arrow[rr, "\Delta\otimes\Delta"] \arrow[d, "\nabla"] & & B \otimes B \otimes B \otimes B \arrow[dd, "\mathrm{id}\otimes\tau\otimes\mathrm{id}"] \\
B \arrow[d, "\Delta"] & & \\
B \otimes B & & B \otimes B \otimes B \otimes B \arrow[ll, "\nabla\otimes\nabla"']
\end{tikzcd}$$

$$\begin{tikzcd}
B \otimes B \arrow[r, "\nabla"] \arrow[d, "\epsilon\otimes\epsilon"] & B \arrow[d, "\epsilon"] \\
\mathbb{k} \otimes \mathbb{k} \arrow[r, "\cong"] & \mathbb{k}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{k} \arrow[r, "\eta"] \arrow[d, "\cong"] & B \arrow[d, "\Delta"] \\
\mathbb{k} \otimes \mathbb{k} \arrow[r, "\eta\otimes\eta"] & B \otimes B
\end{tikzcd}$$

$$\begin{tikzcd}
\mathbb{k} \arrow[r, "\eta"] \arrow[dr, "\mathrm{id}"] & B \arrow[d, "\epsilon"] \\
& \mathbb{k}
\end{tikzcd}$$

Remark 7.1. The relations represented by the diagrams are

$$\sum_{(xy)} (xy)_{(1)} \otimes (xy)_{(2)} = \sum_{(x),(y)} x_{(1)}y_{(1)} \otimes x_{(2)}y_{(2)}$$

$$\epsilon(xy) = \epsilon(x)\epsilon(y)$$

$$\Delta(\lambda) = 1 \otimes \lambda$$

$$\epsilon(\lambda) = \lambda$$

These diagrams are called the compatibility diagrams because they express a compatibility between the algebra structure and the coalgebra structure on B. There are examples of an algebra and coalgebra on the same vector space which are not compatible to form a bialgebra structure.

Lemma 7.2. *Let B have algebra and coalgebra structure. Then the following are equivalent:*

- *The product and unit maps are coalgebra homomorphisms.*
- *The coproduct and counit maps are algebra homomorphisms.*
- *B is a bialgebra.*

We will shorten $(B, \nabla, \Delta, \eta, \epsilon)$ to B when convenient.

Definition (Primitive Element)**.** *A primitive element $x \in B$ is a element satisfying $\Delta(x) = 1 \otimes x + x \otimes 1$ and $\epsilon(x) = 0$.*

Example 7.3 (Field)*.* A field $\Bbbk$ with the previously defined algebra and coalgebra structure is a bialgebra.

Example 7.4 (Polynomial Bialgebra)*.* The polynomial ring $\Bbbk[x]$ with the free algebra and polynomial coalgebra structure is a bialgebra.

Example 7.5 (Monoid Bialgebra)*.* Let M be a monoid. The free vector space $\Bbbk M$ with the monoid algebra and grouplike coalgebra structure is a bialgebra.

Remark 7.6*.* The matrix algebra and coalgebra structures defined previously are not compatible.

8. Hopf Algebra

Definition (Hopf Algebra)**.** *A Hopf algebra $(H, \nabla, \Delta, \eta, \epsilon, s)$ is a bialgebra with a linear antipode map $s : H \to H$ such that the following diagram commutes.*

$$\begin{tikzcd}
H \otimes H \arrow[rr, "s \otimes \mathrm{id}"] & & H \otimes H \arrow[d, "\nabla"] \\
H \arrow[u, "\Delta"] \arrow[r, "\epsilon"] \arrow[d, "\Delta"'] & \Bbbk \arrow[r, "\eta"] & H \\
H \otimes H \arrow[rr, "\mathrm{id} \otimes s"] & & H \otimes H \arrow[u, "\nabla"']
\end{tikzcd}$$

Remark 8.1. The relation represented by the diagram is

$$\sum_{(x)} s(x_{(1)})x_{(2)} = \sum_{(x)} x_{(1)}s(x_{(2)}) = \epsilon(x)$$

or equivalently,

$$s * \mathrm{id} = \mathrm{id} * s = \epsilon$$

Since the convolution algebra on $\mathrm{Hom}(H, H)$ is a ring, if id has a two-sided inverse s, then s is unique. Not all bialgebras are Hopf algebras.

Most of the examples below have been collected by Dascalescu et al. [3].

Example 8.2 (Field). The bialgebra on a field $\Bbbk$ is a Hopf algebra with $s = \mathrm{id}$.

Example 8.3 (Polynomial Hopf Algebra). The polynomial bialgebra on $\Bbbk[x]$ is a Hopf algebra with antipode given by $s(x^n) = (-x)^n$.

Example 8.4 (Divided Power Hopf Algebra). Let $D = \{x^{[n]} : n \in \mathbb{N}_0\}$. Define a Hopf algebra on $\Bbbk D$ satisfying

$$x^{[m]}x^{[n]} = \binom{m+n}{m} x^{[m+n]}$$

$$\Delta(x^{[n]}) = \sum_{j=0}^{n} x^{[j]} \otimes x^{[n-j]}$$

$$\eta(1) = x^{[0]}$$

$$\epsilon(x^{[n]}) = \begin{cases} 1 & n = 0 \\ 0 & n \geq 1 \end{cases}$$

$$s(x^{(n)}) = -\sum_{j=0}^{n-1} s(x^{[j]})x^{[n-j]}$$

If $\mathrm{char}(\Bbbk) = 0$, then $\Bbbk D$ is Hopf algebra isomorphic to $\Bbbk[x]$ under the map which takes $x^{[n]}$ to $\dfrac{x^n}{n!}$. If $\mathrm{char}(\Bbbk) = p$ for a prime p, then this map is undefined at $x^{[p]}$ because it would result in division by $p! = 0$. In either case, $\Bbbk D$ is called the divided power Hopf algebra over $\Bbbk$.

Example 8.5 (Group Hopf Algebra). Let G be a group. The free vector space $\Bbbk G$ is a Hopf algebra with the monoid algebra and grouplike coalgebra structures, and antipode satisfying $s(i(g)) = i(g^{-1})$.

Remark 8.6. Let M be a monoid with at least one non-invertible element x. Then the bialgebra on $\Bbbk M$ is not a Hopf algebra, because the antipode would require $s(i(x))i(x) = i(x)s(i(x)) = 1$. Suppose towards contradiction such an element $s(i(x))$ exists and write it as

$$s(x) = \sum_{m \in M} \lambda_m i(m)$$

But then

$$\sum_{m \in M} \lambda_m i(x) i(m) = \sum_{m \in M} \lambda_m i(xm) = i(1_M)$$
$$\sum_{m \in M} \lambda_m i(m) i(x) = \sum_{m \in M} \lambda_m i(mx) = i(1_M)$$

and yet the projection of at least one of these sums onto the span of $i(1_M)$ is 0 because x is non-invertible. This is a contradiction.

Example 8.7. Let Q be a 2-dimensional $\mathbb{C}$-vector space with basis $\{|0\rangle, |1\rangle\}$. Extend this notation to $Q^{\otimes n}$ by recursively defining

$$|0\rangle \otimes |b\rangle = |0|b\rangle$$

$$|1\rangle \otimes |b\rangle = |1|b\rangle$$

where b is a binary string and $|$ is the concatenation operation. This is the Dirac ket notation in quantum physics.

In a quantum computer, an array of n qubits may have any state in $Q^{\otimes n}$ of the form

$$\sum_{b \in \{0,1\}^n} \alpha_b |b\rangle$$

where

$$\sum_{b \in \{0,1\}^n} |\alpha_b|^2 = 1$$

A quantum gate or circuit is a unitary linear map on an array of qubits ("gate" typically implies fewer qubits and "circuit" implies more). For example, the CNOT gate on 2 qubits satisfies

$$\mathrm{CNOT}(|00\rangle) = |00\rangle$$

$$\mathrm{CNOT}(|01\rangle) = |01\rangle$$

$$\mathrm{CNOT}(|10\rangle) = |11\rangle$$

$$\mathrm{CNOT}(|11\rangle) = |10\rangle$$

Every quantum circuit can be constructed via tensors and compositions of CNOT gates and single-qubit gates [8].

Now, consider the group structure on $\{|0\rangle, |1\rangle\}$ where $|0\rangle$ is the identity. Then Q has an induced group Hopf algebra structure. Sasakura notes that CNOT is equal to the map $(\mathrm{id} \otimes \nabla)(\Delta \otimes \mathrm{id})$. Therefore every quantum circuit may be described in terms of the usual Hopf algebra operations.

Example 8.8 (Tensor Algebra). Let V be a vector space and let $\mathcal{T}V = \bigoplus_{j=0}^{\infty} V^{\otimes j}$. To avoid ambiguity, let $\boxtimes$ be the symbol for the tensor of elements of $\mathcal{T}V$. The tensor algebra on V is $\mathcal{T}V$ with Hopf algebra structure satisfying

$$
\begin{aligned}
\nabla((x_1 \otimes \cdots \otimes x_m) \boxtimes (y_1 \otimes \cdots \otimes y_n)) &= x_1 \otimes \cdots \otimes x_m \otimes y_1 \otimes \cdots \otimes y_n \\
\Delta((x_1 \otimes \cdots \otimes x_n)) &= (x_1 \otimes \cdots \otimes x_n) \boxtimes 1 \\
&\quad + 1 \boxtimes (x_1 \otimes \cdots \otimes x_n) \\
\eta(1) &= 1 \\
\epsilon(1) &= 1 \\
\epsilon(x_1 \otimes \cdots \otimes x_n) &= 0,\ n \geq 1 \\
s(x_1 \otimes \cdots \otimes x_n) &= (-1)^n x_n \otimes \cdots x_1
\end{aligned}
$$

Definition (Hopf Module)**.** *Let H be a Hopf algebra. A right H-Hopf module is a $\Bbbk$-vector space with right H-module and right H-comodule structure such that the following diagram commutes.*

$$
\begin{tikzcd}
M \otimes H \arrow[rr, "c\otimes\Delta"] \arrow[d, "m"] & & M \otimes H \otimes H \otimes H \arrow[dd, "\mathrm{id} \otimes\tau\otimes\mathrm{id}"] \\
M \arrow[d, "c"] & & \\
M \otimes H & & M \otimes H \otimes H \otimes H \arrow[ll, "m\otimes\nabla"]
\end{tikzcd}
$$

Remark 8.9. The relation represented by the diagram are

$$\sum_{(xh)} (xh)_{(0)} \otimes (xh)_{(1)} = \sum_{(x),(h)} x_{(0)}h_{(1)} \otimes x_{(1)}h_{(2)}$$

9. Q-SERIES

We now take a brief detour from algebra and consider q-series. The theory of q-series deals with combinatorial and analytic identities which are deformed by an additional parameter q. Many of the constructions of noncommutative Hopf algebras, such as the Taft algebras, depend on a parameter q in this way.

Definition (q-Shifted Factorial)**.** *The q-shifted factorial $(x;q)_n$ is defined*

$$(x;q)_n = \begin{cases} 1 & n = 0 \\ (1-x)(1-xq)(1-xq^2)\cdots(1-xq^{n-1}) & n \geq 1 \end{cases}$$

Remark 9.1. In particular we have the identity

$$(q;q)_n = \begin{cases} 1 & n = 0 \\ (1-q)(1-q^2)\cdots(1-q^n) & n \geq 1 \end{cases}$$

It can be shown that $\frac{1}{(q;q)_n}$ is the ordinary generating function (in q) for integer partitions whose parts are at most n. There are many other combinatorial formulas related to integer partitions involving the q-shifted factorial.

Definition (q-Binomial Coefficient)**.** *The q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is defined*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}$$

Remark 9.2. Another partition identity is that $\begin{bmatrix} m+n \\ k \end{bmatrix}_q$ is the ordinary generating function of the number of integer partitions with at most m parts each at most n.

Lemma 9.3. *The q-binomial coefficient satisfies*

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ n-k \end{bmatrix}_q$$

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_q = \begin{bmatrix} n \\ n \end{bmatrix}_q = 1$$

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n \\ k \end{bmatrix}_q + \begin{bmatrix} n \\ k-1 \end{bmatrix}_q$$

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k \end{bmatrix}_q + q^{n-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_q$$

Proof. The first identity follows because of the symmetry between k and $n-k$ in the denominator. The second identity follows by direct substitution. The third identity requires a little more work.

$$\begin{aligned}
\begin{bmatrix} n+1 \\ k \end{bmatrix}_q &= \frac{(q;q)_{n+1}}{(q;q)_k(q;q)_{n+1-k}} \\
&= \frac{(q;q)_n(1-q^{n+1})}{(q;q)_k(q;q)_{n-k+1}} \\
&= \frac{(q;q)_n(q^k-q^{n+1})}{(q;q)_k(q;q)_{n-k+1}} + \frac{(q;q)_n(1-q^k)}{(q;q)_k(q;q)_{n-k+1}} \\
&= q^k\frac{(q;q)_n(1-q^{n-k+1})}{(q;q)_k(q;q)_{n-k}(1-q^{n-k+1}} + \frac{(q;q)_n(1-q^k)}{(q;q)_{k-1}(q;q)_{n-k+1}(1-q^k)} \\
&= q^k\begin{bmatrix} n \\ k \end{bmatrix}_q + \begin{bmatrix} n \\ k-1 \end{bmatrix}_q
\end{aligned}$$

The fourth follows from the first and third by substituting $n-k+1$ for k.

$$\begin{aligned}
\begin{bmatrix} n+1 \\ k \end{bmatrix}_q &= \begin{bmatrix} n+1 \\ n-k+1 \end{bmatrix}_q \\
&= q^{n-k+1}\begin{bmatrix} n \\ n-k+1 \end{bmatrix}_q + \begin{bmatrix} n \\ n-k \end{bmatrix}_q \\
&= q^{n-k+1}\begin{bmatrix} n \\ k-1 \end{bmatrix}_q + \begin{bmatrix} n \\ k \end{bmatrix}_q
\end{aligned}$$

□

The following lemma gives the formula which we will use to give an explicit coproduct for the Taft algebra.

Lemma 9.4. *Let x, y be elements of an algebra and $q \in \Bbbk$ such that $yx = qxy$. Then for all $n \in \mathbb{N}$,*

$$(x+y)^n = \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q x^j y^{n-j}$$

Proof. We prove this by induction. For $n = 0$, the left- and right-hand sides are both 1. Suppose the identity holds for a given n. Then

$$\begin{aligned}
(x+y)^{n+1} &= (x+y) \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q x^j y^{n-j} \\
&= \left(\sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q x^{j+1} y^{n-j} \right) + \left(\sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q y x^j y^{n-j} \right) \\
&= \left(\sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q x^{j+1} y^{n-j} \right) + \left(\sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_q q^j x^j y^{n-j+1} \right) \\
&= \left(\sum_{j=1}^{n+1} \begin{bmatrix} n \\ j-1 \end{bmatrix}_q x^j y^{n-j+1} \right) + \left(\sum_{j=0}^{n} q^j \begin{bmatrix} n \\ j \end{bmatrix}_q x^j y^{n-j+1} \right) \\
&= y^{n+1} + x^{n+1} + \sum_{j=1}^{n} \left(\begin{bmatrix} n \\ j-1 \end{bmatrix}_q + q^j \begin{bmatrix} n \\ j \end{bmatrix}_q \right) x^j y^{n-j+1} \\
&= \sum_{j=0}^{n+1} \begin{bmatrix} n+1 \\ j \end{bmatrix}_q x^j y^{n-j+1}
\end{aligned}$$

as desired. ∎

If the reader is interested in the theory of q-series, we refer them to the excellent course notes taken by Heesung Yang [12] at a seminar on q-series given by Mourad Ismail.

10. Sweedler Algebra / Taft Algebra

Moss Sweedler introduced the first examples of Hopf algebras that are neither commutative or cocommutative [3, p. 166]. The better known example is 4-dimensional, but this is in fact a quotient of Sweedler's infinite-dimensional example [10, p. 89].

Definition. *The infinite-dimensional Sweedler algebra is the Hopf algebra on $\Bbbk\langle x, x^{-1}, y\rangle$ given by*

$$\begin{array}{lll} \Delta(x) = x \otimes x & \Delta(x^{-1}) = x^{-1} \otimes x^{-1} & \Delta(y) = y \otimes x + 1 \otimes y \\ \epsilon(x) = 1 & \epsilon(x^{-1}) = 1 & \epsilon(y) = 0 \\ s(x) = x^{-1} & s(x^{-1}) = x & s(y) = -yx^{-1} \end{array}$$

Remark 10.1. This algebra is not commutative because x and y are independent and do not commute. It is not cocommutative because $\Delta(y) = y \otimes x + 1 \otimes y \neq x \otimes y + y \otimes 1$.

Definition (Sweedler Algebra)**.** *The 4-dimensional Sweedler algebra is the quotient of the infinite-dimensional Sweedler algebra by the relations*

$$x^2 = 1$$

$$y^2 = 0$$

$$yx = -xy$$

Remark 10.2. This algebra is not quite commutative because of the negative sign in the final identity. Note that the relation $x^2 = 1$ implies $x = x^{-1}$, so this algebra is also a quotient of $\Bbbk\langle x, y\rangle$. It is also not cocommutative because the additional relations do not modify the definition $\Delta(y) = y \otimes x + 1 \otimes y$.

Earl Taft generalized the 4-dimensional Sweedler algebra in a different direction.

Definition. *Let $n \geq 2$ and assume that $\Bbbk$ contains a primitive n-th root of unity q. The n-th Taft algebra T_n is the quotient of $\Bbbk[x, y]$ by the relations*

$$x^n = 1$$

$$y^n = 0$$

$$yx = qxy$$

T_n extends to a Hopf algebra with coproduct, counit, and antipode satisfying

$$\Delta(x) = x \otimes x \qquad \Delta(y) = y \otimes x + 1 \otimes y$$

$$\epsilon(x) = 1 \qquad \epsilon(y) = 0$$

$$s(x) = x^{n-1} \qquad s(y) = -yx^{n-1}$$

Remark 10.3. The only primitive square root of unity is -1, and indeed T_2 is isomorphic to the 4-dimensional Sweedler algebra. In general, the q-commutativity relation leads to some complicated identities. The following lemma gives the action of Δ on the basis $\{x^j y^k : 0 \leq j < n, 0 \leq y < n\}$.

Lemma 10.4. *Consider the n-th Taft algebra T_n. For all $j, k \leq n$,*

$$\Delta(x^j y^k) = \sum_{\ell=0}^{k} \begin{bmatrix} k \\ \ell \end{bmatrix}_q x^j y^\ell \otimes x^{j+\ell} y^{k-\ell}$$

Proof. Since Δ is an algebra homomorphism, $\Delta(x^j y^k) = (\Delta(x))^j (\Delta(y))^k$. Calculate that $(1 \otimes y)(y \otimes x) = q(y \otimes x)(1 \otimes y)$. By Lemma 9.4,

$$\begin{aligned}
(\Delta(y))^k &= (y \otimes x + 1 \otimes y)^k \\
&= \sum_{\ell=0}^{k} \begin{bmatrix} k \\ \ell \end{bmatrix}_q (y \otimes x)^\ell (1 \otimes y)^{k-\ell} \\
&= \sum_{\ell=0}^{k} \begin{bmatrix} k \\ \ell \end{bmatrix}_q y^\ell \otimes x^\ell y^{k-\ell} \\
\implies \Delta(x^j y^k) &= (x \otimes x)^j \sum_{\ell=0}^{k} \begin{bmatrix} k \\ \ell \end{bmatrix}_q y^\ell \otimes x^\ell y^{k-\ell} \\
&= (x^j \otimes x^j) \sum_{\ell=0}^{k} \begin{bmatrix} k \\ \ell \end{bmatrix}_q y^\ell \otimes x^\ell y^{k-\ell} \\
&= \sum_{\ell=0}^{k} \begin{bmatrix} k \\ \ell \end{bmatrix}_q x^j y^\ell \otimes x^{j+\ell} y^{k-\ell}
\end{aligned}$$

$\square$

11. Research

We are currently working on classification of Nichols algebras using the GAP computer algebra system [5]. A Nichols algebra is a special quotient of the tensor algebra of a Yetter-Drinfeld module. A Yetter-Drinfeld module is not a Hopf module, but a different type of module over a Hopf algebra in a braided monoidal category. Our method mostly consists of running algorithms to find Grobner bases of Nichols algebras in an attempt to find finite generating sets of the quotient relations.

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